Photons and Atoms

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Introduction

Fundamental importance of the atom-field interaction problem

- Provides all information we have on the universe
- Provides the most precise theory so far: QED
- Provides the best tests of fundamental quantum physics
Introduction

The practical importance of the atom-field interaction problem

- Lasers
- Atomic clocks
- Cold atoms and BEC
- Quantum simulation
Outline of this course

Chapter 1: Field quantization

1. Field eigenmodes
2. Quantum field
3. Field quantum states
4. Beamsplitter
5. Field relaxation
Outline of this course

Chapter 2: Atom-field interaction

1. Two-level atom
2. Spontaneous emission
3. Photodetection
4. The dressed atom
Outline of this course

Chapter 3: Cavity Quantum Electrodynamics

1 Experimental tools
2 Resonant CQED
3 Dispersive CQED
4 Perspectives
Prerequisites

I assume a knowledge of:

- Classical electromagnetism (fields, potentials, Maxwell equations in vacuum, field energy and momentum, gauge, plane waves, polarization, dipole radiation, simple cavities...)
- Basic quantum physics (postulates, hermitian algebra, Hamiltonian evolution, density matrix, harmonic oscillator, hydrogen atom...)

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Bibliography

- Schleich, *Quantum optics in phase space*, Wiley 2000
- Meystre and Sargent *Elements of quantum optics*, Springer 1999
- Scully and Zubairy *Quantum optics*, 1997
- Loudon *Quantum theory of light*, OUP 1983
- Haroche and Raimond *Exploring the quantum*, OUP 2006
- and, of course, these lecture handouts.
Online lecture notes

A more extensive version of this course is given at the international master degree of ENS (ICFP). The lecture notes are available on the following sites:

- [www.cqed.org](http://www.cqed.org), following the menu items ‘teaching’, ‘Jean-Michel Raimond’
- [http://www.lkb.upmc.fr/cqed/teachingjmr/](http://www.lkb.upmc.fr/cqed/teachingjmr/)
Chapter 1: Field quantization
How to treat the radiation field in a quantum way?

Outline of the Chapter:

- A brief reminder on the Blackbody problem (Planck 1900)
- Field eigenmodes
- Quantum field
- Field quantum states
- Beamsplitter
- Field relaxation
A brief reminder on the Blackbody problem (Planck 1900)
The Blackbody problem

Compute the spectrum of the equilibrium radiation in an oven at temperature $T$. A mix of thermodynamics and classical electromagnetism.

Simple strategy:

- Count the number of independent modes of the electromagnetic field (a basis of solutions of Maxwell equations with limiting conditions)
- Treat each mode as an harmonic oscillator at equilibrium and attribute to it the average energy $k_b T$.
- Compare to the experimentally well-known spectrum.
Counting the modes

Assume a rectangular volume for the oven (dimensions $L_x$, $L_y$ and $L_z$), with periodic boundary conditions. Supports only plane waves with $\mathbf{k} = (k_x, k_y, k_z)$ so that

$$k_x = \frac{2\pi}{L_x} n_x$$

where $n_{x,y,z}$ is a set of three positive or negative integers. Two orthogonal polarizations for each set of integers. Energies of all these ‘modes’ add up independently (detailed justification later).

$N_\nu$, the total number of modes with $k < 2\pi\nu/c$. Spectral density of modes $\rho_\nu$ is the number of modes per unit volume between $\nu$ and $\nu + d\nu$: $\rho_\nu \, d\nu$

$$\rho_\nu = \frac{1}{V} \frac{dN_\nu}{d\nu}$$
Counting the modes

Counting the modes with a frequency lower than $\nu$ amounts to counting twice (two orthogonal polarizations) the number of points with integer coordinates in a sphere of radius $2\pi\nu/c$:

$$N_\nu = 2 \frac{4\pi}{3} \left( \frac{2\pi\nu}{c} \right)^3 = \frac{8\pi}{3} \frac{\nu^3}{c^3} \mathcal{V}$$

(3)

where $\mathcal{V}$ is the quantization box volume. Hence

$$\rho_\nu = \frac{8\pi}{c^3} \nu^2$$

(4)
Failure of the classical calculation

Rayleigh-Jeans: attribute, according to classical statistical physics, the average thermal energy $k_b T$ to each mode. Predicts a spectral density of energy:

$$u_{\nu} = k_b T \rho_{\nu}$$  \hspace{1cm} (5)

- Fits with observation at low frequencies $h\nu \ll k_b T$
- Absurd at high frequencies: divergence of the spectrum and infinite power
The light quantum

Planck’s hypothesis

The exchanges of energy between field and matter occur as multiples of a fundamental quantum

\[ h\nu \] (6)

where \( h \) is a ‘Hilfeconstant’. Hence \( E = nh\nu \).

Average energy per mode

\[
\overline{E} = h\nu \frac{\sum_{n=0}^{\infty} ne^{-nh\nu/k_bT}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_bT}}
\] (7)

\[
u = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_bT} - 1}
\] (8)

In excellent agreement with experiments if

\[ h = 6.62 \times 10^{-34} \text{ J/s} \] (9)
Conclusions

- For Planck, the energy exchange quantification is only a trick to compute ‘correctly’ the classical statistical properties.
- Einstein in 1905: a much stronger argument for field quantization based on the entropy properties of radiation. Identifies a term in the entropy comparable to that of a gaz of independent particles with energy $h\nu$.
- The main technical difficulty in the calculation is to properly define and identify the classical field modes. This is also the main difficulty for the complete field quantization. The rest is as simple as quantifying a classical harmonic oscillator.
Field eigenmodes
Positive frequency fields

Time Fourier transform of electric field

\[
E(r, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{E}(r, \omega) e^{-i\omega t} d\omega \tag{10}
\]

Since \(E\) is a real field,

\[
\tilde{E}^*(r, \omega) = \tilde{E}(r, -\omega) \tag{11}
\]

Define the ‘positive frequency field’

\[
E^+(r, t) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \tilde{E}(r, \omega) e^{-i\omega t} d\omega \tag{12}
\]

and the ‘negative frequency field’

\[
E^-(r, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \tilde{E}(r, \omega) e^{-i\omega t} d\omega = (E^+(r, t))^* \tag{13}
\]

Hence

\[
E(r, t) = E^+(r, t) + E^-(r, t) \tag{14}
\]
Eigenmodes basis

‘Box’ of limiting conditions with a total volume $\mathcal{V}$ and arbitrary shape. Orthogonal basis for the solutions of Maxwell equations for the electric field (a Hilbert space):

$$f_\ell(r)e^{-i\omega_\ell t}$$ (15)

where the dimensionless amplitude $f_\ell$ is divergence-free and obeys the Helmholtz equation:

$$\Delta f_\ell + \frac{\omega_\ell^2}{c^2} f_\ell = 0$$ (16)

Orthogonality:

$$\int_\mathcal{V} d^3r f_\ell^*(r) \cdot f_{\ell'}(r) = \delta_{\ell,\ell'} \mathcal{V}$$ (17)

Normalization:

$$\int_\mathcal{V} d^3r |f_\ell(r)|^2 = \mathcal{V}$$ (18)
Normalization

We have defined a normalization condition adapted to a fictitious quantization box. We shall use a different condition when there is an actual ‘box’ defining the limit conditions, i.e. in the case of Cavity (or circuit) Quantum Electrodynamics.

In this case, we choose to normalize $f = |f|$ to one at the field maximum. We then define the cavity mode volume $\mathcal{V}$ by

$$\mathcal{V} = \int_{\mathcal{V}} d^3r \ |f_\ell(r)|^2 .$$

(19)

This change of convention has no influence on the following.
Eigenmodes basis

Expand the positive frequency field on this mode basis

\[ E^+(r, t) = \sum_{\ell} E_{\ell}(t)f_{\ell}(r) \tag{20} \]

where

\[ E_{\ell}(t) = \frac{1}{V} \int E^+(r, t) \cdot f^*_{\ell}(r) \, d^3r \tag{21} \]

The amplitude is obviously a harmonic function of time

\[ E_{\ell}(t) = E_{\ell}(0)e^{-i\omega_{\ell}t} \tag{22} \]

Finally

\[ E^+(r, t) = \sum_{\ell} E_{\ell}(0)e^{-i\omega_{\ell}t}f_{\ell}(r) \tag{23} \]
Plane-wave basis

- A simple basis for a rectangular box and periodic boundaries.
- Set of plane waves with
  \[ \mathbf{k}_n = (k_x, k_y, k_z) = (n_x 2\pi/L_x, n_y 2\pi/L_y, n_z 2\pi/L_z), \]
  where the \( n \)'s are positive or negative (not all equal to zero).
- For each \( n = (n_x, n_y, n_z) \), two orthogonal linear polarizations \( \epsilon_1 \) and \( \epsilon_2 \), perpendicular to \( \mathbf{k} \): \( \epsilon_1 \times \epsilon_2 = \mathbf{u}_k \).
- Basis
  \[ f_\ell(\mathbf{r}) = \epsilon_\ell e^{i\mathbf{k}_\ell \cdot \mathbf{r}} \tag{24} \]
  with \( \ell = (n_x, n_y, n_z, \epsilon) \)
- Circular polarization basis
  \[ \epsilon_\pm = \frac{\epsilon_1 \pm i\epsilon_2}{\sqrt{2}} \tag{25} \]
  \[ \epsilon_+ \times \epsilon_- = -i\mathbf{u}_k \tag{26} \]
Normal variables

To perform the standard canonical quantization procedure, we need to identify a simple set of conjugate dynamical variables.

The potential vector $A$ is divergence-free in the Coulomb gauge (used in all these lectures) and $E = -\partial A/\partial t$. $A$ can be thus expanded on the same mode basis as $E$

$$A^+(r, t) = \sum_\ell A_\ell(t)f_\ell(r)$$ (27)

Choose the set of amplitudes $A_\ell(t)$ (harmonic functions of time) as the ‘normal variables’ and separate their real and imaginary parts

$$A_\ell(t) = A_\ell(0)e^{-i\omega t} = x_\ell(t) + ip_\ell(t)$$ (28)

Any idea why we call them $x$ and $p$?
Normal variables

Electric field in terms of the normal variables. From $\mathbf{E}^+ = -\partial \mathbf{A}^+ / \partial t$

$$\mathcal{E}_\ell(t) = -\frac{dA_\ell}{dt} = i\omega_\ell A_\ell$$  \hspace{1cm} (29)

and hence

$$\mathbf{E}^+(\mathbf{r}, t) = \sum_\ell i\omega_\ell A_\ell(t)f_\ell(\mathbf{r})$$  \hspace{1cm} (30)

Magnetic field:

$$\mathbf{B}^+(\mathbf{r}, t) = \sum_\ell A_\ell(t)\mathbf{h}_\ell(\mathbf{r})$$  \hspace{1cm} (31)

where

$$\mathbf{h}_\ell(\mathbf{r}) = \nabla \times f_\ell(\mathbf{r})$$  \hspace{1cm} (32)
Field energy

The total field energy

\[
H = \frac{\varepsilon_0}{2} \int E^2 + \frac{1}{2\mu_0} \int B^2
\]  \hspace{1cm} (33)

must be written in terms of real fields

\[
E = 2\text{Re} E^+ = 2\text{Re} \sum_{\ell} i\omega_\ell A_\ell f_\ell
\]  \hspace{1cm} (34)

Taking into account the mode orthogonality

\[
H = \sum_{\ell} H_\ell
\]  \hspace{1cm} (35)

Remains to evaluate energy of one given mode. Drop index \(\ell\) for the time being.
Field energy

Electric energy. Real part of electric field

\[
E = i\omega [Af - A^*f^*] = -2\omega [xf'' + pf']
\]  

(36)

with

\[
f = f' + if''
\]  

(37)

\[
H_e = 2\omega^2 \epsilon_0 \left[ x^2 \int (f'')^2 + p^2 \int (f')^2 + 2xp \int f' \cdot f'' \right]
\]  

(38)

Magnetic energy.

\[
B = Ah + A^*h^* = 2xh' - 2ph''
\]  

(39)

\[
H_b = \frac{2}{\mu_0} \left[ x^2 \int (h')^2 + p^2 \int (h'')^2 - 2xp \int h' \cdot h'' \right]
\]  

(40)

Similar, but not obviously equal, to the electric energy.
Comparing electric and magnetic energies

Let us start with the integral of \((\mathbf{h}')^2\), with \(\mathbf{h} = \nabla \times \mathbf{f}\). Using

\[
\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})
\]  
(41)

we can write

\[
\nabla \cdot [\mathbf{f}' \times (\nabla \times \mathbf{f}')] = (\nabla \times \mathbf{f}')^2 - \mathbf{f}' \cdot (\nabla \times \nabla \times \mathbf{f}')
\]  
(42)

Using that these fields are divergence-free and with Helmholtz equation:

\[
\nabla \cdot [\mathbf{f}' \times (\nabla \times \mathbf{f}')] = (\mathbf{h}')^2 - \frac{\omega^2}{c^2} (\mathbf{f}')^2
\]  
(43)

Integrating over space:

\[
\int (\mathbf{h}')^2 = \frac{\omega^2}{c^2} \int (\mathbf{f}')^2
\]  
(44)

Similarly

\[
\int (\mathbf{h}'')^2 = \frac{\omega^2}{c^2} \int (\mathbf{f}'')^2
\]  
(45)
Comparing electric and magnetic energies

Let us examine $\int \mathbf{h}' \cdot \mathbf{h}''$. With

$$\nabla \cdot [f' \times (\nabla \times f'')] = (\nabla \times f) \cdot (\nabla \times f'') - f' \cdot (\nabla \times \nabla \times f'')$$

(46)

we get

$$\int \mathbf{h}' \cdot \mathbf{h}'' = \frac{\omega^2}{c^2} \int f' \cdot f''$$

(47)

Hence

$$H_b = 2\omega^2 \epsilon_0 \left[ x^2 \int (f')^2 + p^2 \int (f'')^2 - 2xp \int f' \cdot f'' \right]$$

(48)

Using

$$\int (f')^2 + \int (f'')^2 = \mathcal{V}$$

(49)

we get finally

$$H = 2\omega^2 \epsilon_0 \mathcal{V} [x^2 + p^2]$$

(50)
The total energy of the radiation field is thus:

\[ H = \sum_{\ell} H_\ell = \sum_{\ell} 2\omega_\ell^2 \epsilon_0 V \left[ x_\ell^2 + p_\ell^2 \right] \]  

(51)

A collection of independent harmonic oscillators.
Canonical variables

- Need canonically conjugate variables for quantization: $x_c$ and $p_c$ such that

\[
\frac{dx_c}{dt} = \frac{\partial H}{\partial p_c} \quad \text{and} \quad \frac{dp_c}{dt} = -\frac{\partial H}{\partial x_c}
\] (52)

- $x$ and $p$ are not canonical, since

\[
\frac{dx}{dt} = \omega p \neq \frac{\partial H}{\partial p} = 4\omega^2 \varepsilon_0 \mathcal{V} p
\] (53)

- Canonical amplitude

\[
\alpha(t) = 2\sqrt{\varepsilon_0 \omega \mathcal{V}} A(t)
\] (54)

- Canonical position and momentum:

\[
\alpha(t) = x_c + ip_c
\] (55)

i.e.

\[
x_c = 2\sqrt{\varepsilon_0 \omega \mathcal{V}} x \quad \text{and} \quad p_c = 2\sqrt{\varepsilon_0 \omega \mathcal{V}} p
\] (56)
Canonical variables

Mode energy

\[ H = \frac{\omega}{2} \left( x_c^2 + p_c^2 \right) \]  

and obviously

\[ \frac{dx_c}{dt} = \frac{\partial H}{\partial p_c} \quad \text{and} \quad \frac{dp_c}{dt} = -\frac{\partial H}{\partial x_c} \]  

Proper canonical variables for the canonical quantization procedure.

Note that the \( x_c \) and \( p_c \) coordinates are not dimensionless (their joint dimension is the square root of an action)
Field momentum

Density of momentum proportional to the Poynting vector

\[ g = \frac{\Pi}{c^2} \quad \text{with} \quad \Pi = \frac{E \times B}{\mu_0} \]  

(59)

The plane wave mode basis is most convenient to describe the momentum

\[ E^+(r, t) = \sum_\ell E^+_{\ell} = \sum_\ell i\omega_{\ell} A_{\ell}(t) \epsilon_{\ell} e^{ik_{\ell} \cdot r} \]  

(60)

and

\[ B^+(r, t) = \sum_\ell B^+_{\ell} = \sum_\ell A_{\ell}(t)(ik_{\ell} \times \epsilon_{\ell}) e^{ik_{\ell} \cdot r} \]  

(61)
Field momentum

Using orthogonality of modes and polarizations

\[ \mathbf{P} = \sum_{\ell} \mathbf{P}_{\ell} \quad (62) \]

with

\[ \mathbf{P}_{\ell} = \varepsilon_0 \int (\mathbf{E}^+_{\ell} + \mathbf{E}^-_{\ell}) \times (\mathbf{B}^+_{\ell} + \mathbf{B}^-_{\ell}) \quad (63) \]

After a painful calculation

\[ \mathbf{P}_{\ell} = 2\varepsilon_0 V_\omega |A_{\ell}|^2 \varepsilon_{\ell} \times (\mathbf{k}_{\ell} \times \varepsilon_{\ell}) \quad (64) \]

or, finally

\[ \mathbf{P} = \frac{1}{2} \sum_{\ell} |\alpha_{\ell}|^2 \mathbf{k}_{\ell} \quad (65) \]

with a clear interpretation.
Field Angular momentum

Angular momentum density $\mathbf{r} \times \mathbf{g}$ and hence

$$\mathbf{J} = \epsilon_0 \int \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \, d^3\mathbf{r}$$  \hspace{1cm} (66)

A difficult calculation leads to

$$\mathbf{J} = \mathbf{L} + \mathbf{S},$$  \hspace{1cm} (67)

where

$$\mathbf{S} = \epsilon_0 \int \mathbf{E} \times \mathbf{A} \, d^3\mathbf{r}$$  \hspace{1cm} (68)

is the field’s ‘intrinsic angular momentum’ and

$$\mathbf{L} = \epsilon_0 \int d^3\mathbf{r} \sum_j E_j (\mathbf{r} \cdot \nabla) A_j, \quad j = (x, y, z)$$  \hspace{1cm} (69)

is the field’s ‘orbital angular momentum’.
Spin angular momentum

Plane wave basis with circular polarizations

\[ \mathbf{S} = i\epsilon_0 \mathcal{V} \sum_n \omega_n \left[ \mathcal{A}_{n+}\mathcal{A}_{n+}^*(\epsilon_+ \times \epsilon_+) + \mathcal{A}_{n-}\mathcal{A}_{n-}^*(\epsilon_- \times \epsilon_-) - \text{c.c.} \right] \]  \hspace{1cm} (70)

Using \( \epsilon_+ \times \epsilon_+^* = \epsilon_+ \times \epsilon_- = -i\mathbf{u}_k \) and \( \epsilon_- \times \epsilon_-^* = i\mathbf{u}_k \)

\[ \mathbf{S} = \frac{1}{2} \sum_n \left[ |\alpha_{n+}|^2 - |\alpha_{n-}|^2 \right] \mathbf{u}_k \]  \hspace{1cm} (71)

with an equally simple interpretation.
Quantum field
Field quantization

The field is a collection of independent harmonic oscillators. Let us quantify all of them independently, using the Dirac approach. Since we will be mostly considering a single mode, we frequently drop the mode index in the following.

The conjugate classical variables $x_c$ and $p_c$ should be replaced by two operators $X$ and $P$ (position and momentum operators, dimension also the square root of an action) acting in an infinite dimension Hilbert space, with the commutation rule:

$$[X, P] = i\hbar$$  \hspace{1cm} (72)
Annihilation and creation operators

\[ a = \frac{1}{\sqrt{2\hbar}}(X + iP) \]  (73)

and

\[ a^\dagger = \frac{1}{\sqrt{2\hbar}}(X - iP) \]  (74)

with

\[ [a, a^\dagger] = 1 \]  (75)

Note that \( a \) and \( a^\dagger \) are dimensionless. The reverse relations read

\[ X = \sqrt{\frac{\hbar}{2}} \left( a + a^\dagger \right) \]  (76)

and

\[ P = i\sqrt{\frac{\hbar}{2}} \left( a^\dagger - a \right) \]  (77)
Field quadratures

Define reduced units

\[ X_0 = \frac{X}{\sqrt{2\hbar}} \quad \text{and} \quad P_0 = \frac{P}{\sqrt{2\hbar}} \]  

(78)

With these definitions

\[ [X_0, P_0] = \frac{i}{2} \]  

(79)

\[ a = X_0 + iP_0, \quad a^\dagger = X_0 - iP_0, \quad X_0 = \frac{a + a^\dagger}{2}, \quad P_0 = \frac{i(a^\dagger - a)}{2} \]  

(80)
Chapter 1: Field quantization

Quantum field

Hamiltonian

\[ H = \frac{\omega}{2}(X^2 + P^2) = \hbar \omega (X_0^2 + P_0^2) \]  

or

\[ H = \frac{\hbar \omega}{4} \left[ (a + a^\dagger)^2 - (a^\dagger - a)^2 \right] \]  

and, in the ‘normal order’,

\[ H = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right) \]  

whose diagonalization is described in all textbooks.
Number operator

\[ N = a^\dagger a \quad (84) \]

Commutation relations:

\[ [N, a] = -a \quad \text{and} \quad [N, a^\dagger] = a^\dagger \quad (85) \]

Eigenvalues: all positive integers, with nondegenerate eigenstates

\[ N |n\rangle = n |n\rangle, \quad (86) \]

Hence, the eigenenergies are

\[ E_n = \left( n + \frac{1}{2} \right) \hbar \omega \quad (87) \]

Ground state: ‘vacuum’, |0\rangle, energy \( \hbar \omega /2 \)
Fock states

The states $|n\rangle$ are the ‘photon number states’ with the orthogonality relation

$$\langle n | p \rangle = \delta_{n,p} \tag{88}$$

Annihilation and creation of photons:

$$a |n\rangle = \sqrt{n} |n - 1\rangle \tag{89}$$

with

$$a |0\rangle = 0 \tag{90}$$

and, similarly

$$a^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle \tag{91}$$

Hence

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \tag{92}$$
All modes

\[ H |n_1, \ldots, n_\ell \ldots \rangle = E_n |n_1, \ldots, n_\ell \ldots \rangle \quad (93) \]

with

\[ E_n = \sum_\ell \left( n_\ell \hbar \omega_\ell + \frac{\hbar \omega_\ell}{2} \right) \quad (94) \]

and

\[ |n_1, \ldots, n_\ell \ldots \rangle = \prod_\ell \frac{(a^{\dagger}_\ell)^{n_\ell}}{\sqrt{n_\ell!}} |0\rangle \quad (95) \]

Note that the vacuum state has an infinite energy (more on that later).
Vector potential operator

Classical normal variables:

$$\mathcal{A} = \frac{1}{2\sqrt{\epsilon_0 \omega}} (x_c + ip_c)$$  \hspace{1cm} (96)

Corresponding quantum operators

$$A_\ell = \frac{1}{2\sqrt{\epsilon_0 \omega_\ell \mathcal{V}}} (X_\ell + iP_\ell) = \sqrt{\frac{\hbar}{2\epsilon_0 \omega_\ell \mathcal{V}}} a_\ell$$ \hspace{1cm} (97)

Positive frequency vector potential

$$\mathbf{A}^+(\mathbf{r}) = \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0 \omega_\ell \mathcal{V}}} a_\ell \mathbf{f}_\ell(\mathbf{r})$$ \hspace{1cm} (98)

Hermitian vector potential:

$$\mathbf{A}(\mathbf{r}) = \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0 \omega_\ell \mathcal{V}}} \left( a_\ell \mathbf{f}_\ell(\mathbf{r}) + a_\ell^\dagger \mathbf{f}_\ell^*(\mathbf{r}) \right)$$ \hspace{1cm} (99)
Electric and magnetic field operators

The hermitian electric field is:

\[ E(r) = i \sum_{\ell} \mathcal{E}_{\ell} \left( a_{\ell} f_{\ell}(r) - a_{\ell}^\dagger f_{\ell}^*(r) \right) \]  \hspace{1cm} (100)

where we define the ‘field per photon in mode \( \ell \)’ by

\[ \mathcal{E}_{\ell} = \sqrt{\frac{\hbar \omega_{\ell}}{2\varepsilon_0 V}} \]  \hspace{1cm} (101)

Similarly:

\[ B(r) = \sum_{\ell} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_{\ell} V}} \left( a_{\ell} h_{\ell}(r) + a_{\ell}^\dagger h_{\ell}^*(r) \right) \]  \hspace{1cm} (102)

with \( h_{\ell} = \nabla \times f_{\ell} \)
Plane wave mode basis

\[ A^+(r) = \sum_{\ell} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega \ell V}} a_{\ell} \varepsilon_{\ell} e^{i k_{\ell} \cdot r} \]  \hspace{1cm} (103)

\[ E^+(r) = i \sum_{\ell} \varepsilon_{\ell} a_{\ell} \varepsilon_{\ell} e^{i k_{\ell} \cdot r} \]  \hspace{1cm} (104)

\[ B^+(r) = \sum_{\ell} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega \ell V}} a_{\ell} (i k_{\ell} \times \varepsilon_{\ell}) e^{i k_{\ell} \cdot r} \]  \hspace{1cm} (105)
Heisenberg picture

Evolution of annihilation operator

\[ i\hbar \frac{da_H}{dt} = [a_H, H] \quad \text{i.e.} \quad \frac{da_H}{dt} = -i\omega a_H \] (106)

whose immediate solution is

\[ a_H(t) = a_H(0)e^{-i\omega t} = ae^{-i\omega t} \] (107)
Momentum, angular momentum

- Total momentum by replacing $|\alpha_\ell|^2$ in the classical expression by $\alpha_\ell^* \alpha_\ell$ and $\alpha_\ell$ by $a_\ell \sqrt{2\hbar}$

\[
P = \sum_\ell \hbar k_\ell a_\ell^\dagger a_\ell
\]  \hspace{1cm} (108)

- Similarly

\[
S = \sum_n \hbar u_{k_n} [N_{n+} - N_{n-}]
\]  \hspace{1cm} (109)
Field quadratures

Eigenstates of the quadratures:

\[ X_0 |x\rangle = x |x\rangle \quad \text{and} \quad P_0 |p\rangle = p |p\rangle \quad (110) \]

Wavefunctions:

\[ \psi(x) = \langle x | \psi \rangle \quad (111) \]

For the vacuum:

\[ \psi_0(x) = \left(\frac{2}{\pi}\right)^{1/4} e^{-x^2} \quad (112) \]

Also in the \(|p\rangle\) representation:

\[ \tilde{\psi}_0(p) = \left(\frac{2}{\pi}\right)^{1/4} e^{-p^2} \quad (113) \]

Suggests a pictorial representation of the vacuum as a circle with radius one centered at the origin in phase plane.
Field quadratures

For the Fock state $|n\rangle$:

$$\psi_n(x) = \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-x^2} H_n(x\sqrt{2})$$  \hspace{1cm} (114)$$

where $H_n$ is the $n$th Hermite polynomial defined by

$$H_n(u) = (-1)^n e^{u^2} \frac{d^n}{du^n} e^{-u^2}$$  \hspace{1cm} (115)$$

These wavefunctions have $n$ nodes and a a parity $(-1)^n$. 
Field quadratures

General field quadratures

\[ X_{\phi} = \frac{a e^{-i\phi} + a^\dagger e^{i\phi}}{2} \]  

(116)

with obviously \( P_0 = X_{\pi/2} \). Commutation:

\[ [X_{\phi}, X_{\phi+\pi/2}] = \frac{i}{2} \]  

(117)

Heisenberg relations

\[ \Delta X_{\phi}\Delta X_{\phi+\pi/2} \geq \frac{1}{4} \]  

(118)

Eigenstates \( X_{\phi} |x_{\phi}\rangle = x_{\phi} |x_{\phi}\rangle \) with

\[ |x_{\phi+\pi/2}\rangle = \frac{1}{\sqrt{\pi}} \int dy_{\phi} e^{2ix_{\phi+\pi/2}y_{\phi}} |y_{\phi}\rangle \]  

(119)
Field states
Fock states

A basis of the Hilbert space

$$|\psi\rangle = \sum_n c_n |n\rangle$$  \hspace{1cm} (120)

Photon number distribution

$$p_n = |c_n|^2$$  \hspace{1cm} (121)

Mean number of photons

$$\bar{n} = \sum_n n p_n$$  \hspace{1cm} (122)

Photon number variance

$$\Delta N^2 = \langle N^2 \rangle - \langle N \rangle^2 = \sum_n (n - \bar{n})^2 p_n$$  \hspace{1cm} (123)
Fock states

Statistical mixtures

\[ \rho = \sum_{n,p} \rho_{np} \left| n \right\rangle \left\langle p \right| \quad \text{(124)} \]

Photon number distribution

\[ \rho_{nn} = p_n \quad \text{(125)} \]

Fock states are very non-classical

- A large energy
- Zero average fields and potentials since \( \langle n | a | n \rangle = 0 \)

Can we find more intuitive field states? Yes: Coherent states.
Displacement operator

A unitary defined by:

\[ D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \quad (126) \]

where \( \alpha \) is an arbitrary complex amplitude

\[ \alpha = \alpha' + i \alpha'' \quad (127) \]

\[ D(\alpha)^\dagger D(\alpha) = 1 \quad (128) \]

and

\[ D(\alpha)^\dagger = D(-\alpha) \quad (129) \]
Displacement operator

An equivalent expression

\[ D(\alpha) = e^{2i\alpha''}X_0e^{-2i\alpha'P_0} \] (130)

Using the Glauber relation

\[ e^Ae^B = e^{A+B}e^{[A,B]/2} \] (131)

valid when

\[ [A, [A, B]] = [B, [A, B]] = 0 \] (132)

\[ D(\alpha) = e^{-i\alpha'\alpha''}e^{2i\alpha''}X_0e^{-2i\alpha'P_0} \] (133)

a product of displacement operators:

\[ e^{-2i\alpha'P_0} \langle x | = | x + \alpha' \rangle \] (134)

\[ e^{2i\alpha''X_0} \langle p | = | p + \alpha'' \rangle \] (135)
Combination of displacements

Using Glauber

\[ D(\alpha)D(\beta) = e^{(\alpha\beta^* - \alpha^*\beta)/2}D(\alpha + \beta) \]  

(136)

Note that

\[ \Phi = (\alpha\beta^* - \alpha^*\beta)/2i = \frac{\alpha''\beta' - \alpha'\beta''}{2} \]  

(137)

is the surface of the triangle with sides \( \alpha \) and \( \beta \).
Displacement of annihilation operator

Compute $D(-\alpha)aD(\alpha)$. Use Baker-Hausdorff lemma

$$e^A a e^{-A} = a + [A, a] + \frac{1}{2!} [A, [A, a]] + \ldots$$

(138)

for $A = -\alpha a^\dagger + \alpha^* a$, with $[A, a] = \alpha$. Hence

$$D(-\alpha)aD(\alpha) = a + \alpha \mathbb{1}$$

(139)
Coherent states

The coherent states are defined as

$$|\alpha\rangle = D(\alpha)|0\rangle \quad (140)$$

Note that $|0\rangle$ is a coherent state. Coherent states in general are the vacuum displaced by the complex amplitude $\alpha$. Pictorially represented as a radius one circle at the tip of the classical amplitude $\alpha$.

Wavefunction of a coherent state in the $X_0$ representation:

$$\psi_\alpha(x) \propto e^{- (x-\alpha')^2} \quad (141)$$

and in the $P_0$ representation:

$$\tilde{\psi}_\alpha(p) \propto e^{- (p-\alpha'')^2} \quad (142)$$
Coherent states properties

- Right-eigenstates of the annihilation operator

\[ a |\alpha\rangle = aD(\alpha) |0\rangle = D(\alpha)D(-\alpha)aD(\alpha) |0\rangle = (a + \alpha \mathbb{1}) |0\rangle = \alpha |\alpha\rangle \]  \hspace{1cm} (143)

since \( a |0\rangle = 0 \). Hence

\[ \langle \alpha | a |\alpha\rangle = \alpha \quad \text{and} \quad \langle \alpha | a^\dagger |\alpha\rangle = \alpha^* \]  \hspace{1cm} (144)

- Field operators have nonzero eigenvalues in the coherent states:

\[ \langle E \rangle = i\mathcal{E} \left( f(r)\alpha - f^*(r)\alpha^* \right) \]  \hspace{1cm} (145)

\[ \langle A \rangle = \frac{\mathcal{E}}{\omega} \left( f(r)\alpha + f^*(r)\alpha^* \right) \]  \hspace{1cm} (146)
Coherent states properties

- Average photon number

\[ \bar{n} = \langle \alpha | a^\dagger a | \alpha \rangle = |\alpha|^2 \]  \hspace{1cm} (147)

- Photon number variance. Using \( N^2 = a^\dagger aa^\dagger a = (a^\dagger)^2 a^2 + a^\dagger a \)

\[ \langle N^2 \rangle = |\alpha|^4 + |\alpha|^2 \]  \hspace{1cm} (148)

and

\[ \Delta N^2 = |\alpha|^2 = \bar{n} \]  \hspace{1cm} (149)

\[ \frac{\Delta N}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}} \]  \hspace{1cm} (150)
Coherent states properties

- Expansion on the Fock state basis

\[
D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a}
\]  
(151)

with \( a|0\rangle = 0 \):

\[
|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle
\]  
(152)

Expand exponential:

\[
|\alpha\rangle = \sum_n c_n |n\rangle ,
\]  
(153)

with

\[
c_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}
\]  
(154)
Coherent states properties

- Photon number distribution: Poisson statistics

\[ p_n = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = e^{-\bar{n}} \frac{\bar{n}^n}{n!} \]  (155)

For large average photon numbers, reduces to Gaussian statistics

\[ p_n \propto e^{-(n-\bar{n})^2/\bar{n}} \]  (156)

- Scalar product of coherent states

\[
\langle \alpha | \beta \rangle = e^{-(|\alpha|^2+|\beta|^2)/2} \sum_{n,p} \frac{(\alpha^*)^n \beta^p}{\sqrt{n!p!}} \langle n | p \rangle \\
= e^{-(|\alpha|^2+|\beta|^2)/2} e^{\alpha^* \beta} \]  (157)

Square modulus

\[ |\langle \alpha | \beta \rangle|^2 = e^{-|\alpha-\beta|^2} \]  (158)
Coherent states properties

- Overcomplete basis

\[ 1 = \frac{1}{\pi} \int d^2 \alpha \, |\alpha\rangle \langle \alpha| \]  \hspace{1cm} (159)

Leave demonstration as exercise 4. The expansion of a state over an overcomplete basis is not uniquely defined.

- Evolution

\[ |\psi(0)\rangle = |\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \]  \hspace{1cm} (160)

\[ |\psi(t)\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-in\omega t} e^{-i\omega t/2} |n\rangle \]

\[ = e^{-i\omega t/2} |\alpha e^{-i\omega t}\rangle \]  \hspace{1cm} (161)

Evolution of the amplitude is the same as in classical physics

\[ \alpha(t) = \alpha(0)e^{-i\omega t} \]  \hspace{1cm} (162)
Coherent states generation

Coupling a quantum mode with a classical current (model of electronic source or laser). Consider only the field mode at resonance with the source.

\[ j(r, t) = j_0(r) e^{-i\omega_0 t} \]  

From classical interaction energy, \(-j \cdot A\), guess the interaction Hamiltonian

\[ H_i = -\int_V j(r, t) \cdot A(r, t) \, d^3r \approx \sqrt{\frac{\hbar V}{2\epsilon_0 \omega_\ell}} J_0 a_\ell^\dagger + \text{h.c.} \]  

(with the RWA), which obviously generates a displacement operator. The amplitude of the generated coherent state increases linearly with time and the photon number quadratically. A consequence of bosonic amplification.
Seeks an analogue of the classical phase space distributions $f(x, p)$ of statistical physics allowing us to compute any average by

$$\bar{\sigma} = \int f(x, p) \sigma(x, p) \, dx \, dp \quad (165)$$

Transpose that to a field statistical mixture defined by the density operator $\rho$. 
Characteristic functions

Three operators ordering:

- Normal: $a$ on right. e.g. number operator $a^\dagger a$
- Symmetric e.g. $(aa^\dagger + a^\dagger a)$
- Anti-Normal e.g. $aa^\dagger$

Any operator expression can be put in one of these forms by proper commutations of creation and annihilation operators.

Leads to three characteristic functions characterizing $\rho$
Symmetric characteristic function

- Symmetric characteristic function

\[ C_s^{[\rho]}(\lambda) = \langle D(\lambda) \rangle = \text{Tr} \left[ \rho e^{\lambda a^\dagger - \lambda^* a} \right] \tag{166} \]

with

\[ C_s^{[\rho]}(0) = \text{Tr}(\rho) = 1. \tag{167} \]

\(D\) being unitary, all its eigenvalues have a unit modulus. Hence

\[ |C_s^{[\rho]}(\lambda)| \leq 1 \tag{168} \]

and

\[ C_s^{[\rho]}(-\lambda) = \left[ C_s^{[\rho]}(\lambda) \right]^* \tag{169} \]

For a pure state

\[ C_s^{[\psi\rangle\langle\psi]} = \langle \psi | D(\lambda) | \psi \rangle \tag{170} \]
Normal and anti-normal characteristic functions

- Normal characteristic function

\[ C_n^{[\rho]}(\lambda) = \text{Tr} \left[ \rho e^{\lambda a^\dagger} e^{-\lambda^* a} \right] \quad (171) \]

- Anti-normal characteristic function

\[ C_{an}^{[\rho]}(\lambda) = \text{Tr} \left[ \rho e^{-\lambda^* a} e^{\lambda a^\dagger} \right] \quad (172) \]

- Relations (from Glauber formula)

\[ C_n^{[\rho]}(\lambda) = e^{\lambda^2 / 2} C_s^{[\rho]}(\lambda) \quad C_{an}^{[\rho]}(\lambda) = e^{-\lambda^2 / 2} C_s^{[\rho]}(\lambda) \quad (173) \]
The Husimi-\(Q\) representation

Definition: two-dimensional Fourier transform of \(C_{an}^{[\rho]}(\lambda)\)

\[
Q^{[\rho]}(\alpha) = \frac{1}{\pi^2} \int d^2 \lambda \ e^{(\alpha \lambda^* - \alpha^* \lambda)} C_{an}^{[\rho]}(\lambda)
\]  

(174)

After some algebra (see Exercise 5):

\[
Q^{[\rho]}(\alpha) = \frac{1}{\pi} \text{Tr} [\rho \ |\alpha\rangle \langle \alpha|] = \frac{1}{\pi} \langle \alpha| \rho |\alpha\rangle = \frac{1}{\pi} \text{Tr}[|0\rangle \langle 0| D(-\alpha) \rho D(\alpha)]
\]  

(175)

The \(Q\) distribution is positive, bounded by \(1/\pi\) and normalized \((\int d^2 \alpha \ Q(\alpha) = 1)\).
Phase space representations

The Husimi-$Q$ representation

(a) Coherent state $|\beta\rangle$, with $\beta = \sqrt{5}$. (b) Five-photon Fock state. (c) Schrödinger cat state, superposition of two coherent fields $|\pm\beta\rangle$, with $\beta = \sqrt{5}$. (d) Statistical mixture of the same coherent components.
The Wigner function

Definition: Fourier transform of the symmetric characteristic function.

\[
W(\alpha) = \frac{1}{\pi^2} \int d^2 \lambda \ C_s(\lambda) e^{\alpha \lambda^* - \alpha^* \lambda} \tag{176}
\]

After a long derivation:

\[
W(x, p) = \frac{2}{\pi} \text{Tr}[D(-\alpha) \rho D(\alpha) \mathcal{P}] \tag{177}
\]

where the unitary parity operator \( \mathcal{P} \) is defined by

\[
\mathcal{P} |x\rangle = |-x\rangle ; \quad \mathcal{P} |p\rangle = |-p\rangle ; \quad \mathcal{P} |n\rangle = (-1)^n |n\rangle ; \quad \mathcal{P} = e^{i \pi a^+ a} \tag{178}
\]

The module of the average value of \( \mathcal{P} \) is lower than or equal to one and hence:

\[
-2/\pi \leq W(\alpha) \leq 2/\pi \tag{179}
\]
Marginals of the Wigner distribution

\[ P(x) = \langle x | \rho | x \rangle = \int dp \ W(x, p) \] (180)

\[ P(p) = \langle p | \rho | p \rangle = \int dx \ W(x, p) \] (181)

More generally,

\[ P(p_\phi) = \int dx_\phi \ W(x_\phi, p_\phi) \] (182)

with

\[ x_\phi = x \cos \phi + p \sin \phi ; \quad p_\phi = -x \sin \phi + p \cos \phi \] (183)

Moreover, the average of any operator can be directly obtained from the Wigner function

\[ \langle O \rangle = \int dx dp \ W(x, p) o_s(x, p) \] (184)

where \( o_s \) is the symmetrized form of the operator \( O \) in terms of the field quadratures.
Wigner functions: classical states

(a) Vacuum state. (b) Coherent state with $\beta = \sqrt{5}$. (c) Thermal field with $n_{\text{th}} = 1$ photon on the average. (d) A squeezed vacuum state, with a squeezing parameter $\xi = 0.5$. 
Wigner functions: Fock states

Wigner function of a five-photon Fock state.
Wigner functions: cat states

Wigner functions of even (a) and odd (b) 10-photon $\pi$-phase cats. The Wigner function provides a clear depiction of the non-classical features of a quantum state.
Chapter 1: Field quantization

Beamsplitter
Coupling field modes

A simple model for coupling two modes of the radiation field
Chapter 1: Field quantization

Classical approach

Transformation of the electric field amplitudes

\[
\begin{pmatrix}
E'_a \\
E'_b
\end{pmatrix} = U_c \begin{pmatrix}
E_a \\
E_b
\end{pmatrix} = \begin{pmatrix}
t(\omega) & r(\omega) \\
r(\omega) & t(\omega)
\end{pmatrix} \begin{pmatrix}
E_a \\
E_b
\end{pmatrix}
\]

(185)

where the unitary $U_c$ can also be written in a simple case as

\[
U_c(\theta) = \begin{pmatrix}
\cos(\theta/2) & i \sin(\theta/2) \\
i \sin(\theta/2) & \cos(\theta/2)
\end{pmatrix}
\]

(186)
Hamiltonian model

Model the beamsplitter action as a transient application of the Hamiltonian

\[ H_{ab}(t) = -\hbar \frac{g(t)}{2} (ab^\dagger + a^\dagger b) \]  

\( a \) and \( b \): annihilation operators; \( g(t) \) slowly varying real function
Heisenberg point of view

Transformation of the annihilation operator:

\[ a' = U^\dagger a U \quad (188) \]

where

\[ U = e^{-(i/\hbar) \int H_{ab}(t) \, dt} = e^{-iG\theta/2} \quad (189) \]

with

\[ G = -(ab^\dagger + a^\dagger b) \quad \text{and} \quad \theta = \int g(t) \, dt \quad (190) \]

Using Baker-Hausdorff

\[
a' = U^\dagger a U = e^{iG\theta/2} a e^{-iG\theta/2} = a + \frac{i\theta}{2} [G, a] \\
+ \frac{i^2\theta^2}{2!2^2} [G, [G, a]] + \cdots + \frac{i^n\theta^n}{n!2^n} [G, [G, [\cdots, [G, a]]]] + \cdots (191)\]

Heisenberg point of view

With \([G, a] = b\) and \([G, [G, a]] = a\), series sum up to

\[
a' = U^\dagger a U = \cos(\theta/2) a + i \sin(\theta/2) b
\]

and similarly:

\[
b' = U^\dagger b U = i \sin(\theta/2) a + \cos(\theta/2) b
\]

Noting that \(U^\dagger(\theta) = U(-\theta)\)

\[
U a^\dagger U^\dagger = \cos(\theta/2) a^\dagger + i \sin(\theta/2) b^\dagger ; \quad U b^\dagger U^\dagger = i \sin(\theta/2) a^\dagger + \cos(\theta/2) b^\dagger
\]
State transformations

Transformation of some simple states:

- No photon: \( |\Psi\rangle = |0, 0\rangle \). This state is obviously invariant
- One photon in mode \( a \)

\[
U |1, 0\rangle = U a^\dagger |0, 0\rangle = U a^\dagger U^\dagger U |0, 0\rangle = U a^\dagger U^\dagger |0, 0\rangle
\]

(195)

and, using the Heisenberg point of view results in:

\[
U |1, 0\rangle = \left[ \cos(\theta/2) a^\dagger + i \sin(\theta/2) b^\dagger \right] |0, 0\rangle
= \cos(\theta/2) |1, 0\rangle + i \sin(\theta/2) |0, 1\rangle
\]

(196)

- One photon in mode \( b \)

\[
U |0, 1\rangle = \left[ i \sin(\theta/2) a^\dagger + \cos(\theta/2) b^\dagger \right] |0, 0\rangle
= i \sin(\theta/2) |1, 0\rangle + \cos(\theta/2) |0, 1\rangle
\]

(197)
State transformations

- $n$ photons

$$U | n, 0 \rangle = U \frac{(a^\dagger)^n}{\sqrt{n!}} | 0, 0 \rangle = \frac{1}{\sqrt{n!}} U (a^\dagger)^n U^\dagger U | 0, 0 \rangle \quad (198)$$

With $U (a^\dagger)^n U^\dagger = (U a^\dagger U^\dagger)^n$,

$$U | n, 0 \rangle = \frac{1}{\sqrt{n!}} \left[ \cos \frac{\theta}{2} a^\dagger + i \sin \frac{\theta}{2} b^\dagger \right]^n | 0, 0 \rangle \quad (199)$$

expansion of the r.h.s.

$$U | n, 0 \rangle = \sum_{p=0}^{n} \binom{n}{p}^{1/2} [\cos(\theta/2)]^{n-p} [i \sin(\theta/2)]^p | n - p, p \rangle \quad (200)$$
State transformations

- \( n \) photons, balanced splitter (\( \theta = \pi/2 \))

\[
U(\pi/2, 0) | n, 0 \rangle = \frac{1}{\sqrt{2^n}} \sum_{p=0}^{n} \binom{n}{p}^{1/2} (i)^p | n - p, p \rangle
\]  

- Random output selection for each photon
- A massively entangled state of the two output modes
State transformations

- Coherent state $|\alpha\rangle$

$$U |\alpha, 0\rangle = UD_a(\alpha) U^\dagger |0, 0\rangle \hspace{1cm} (202)$$

rewrites, with $Uf(A) U^\dagger = f(UA U^\dagger)$

$$UD(\alpha) U^\dagger = e^{\alpha Ua^\dagger U^\dagger - \alpha^* Ua U^\dagger} \hspace{1cm} (203)$$

and

$$U |\alpha, 0\rangle = D_a [\alpha \cos(\theta/2)] D_b [i\alpha \sin(\theta/2)] |0, 0\rangle \hspace{1cm} (204)$$

finally

$$U |\alpha, 0\rangle = |\alpha \cos(\theta/2), i\alpha \sin(\theta/2)\rangle \hspace{1cm} (205)$$

An unentangled states, with two coherent amplitudes split according to the classical laws.
State transformations

- Photon collision on a beamsplitter

\[
U |1, 1\rangle = Ua^\dagger b^\dagger |0, 0\rangle = Ua^\dagger U^\dagger Ub^\dagger U^\dagger |0, 0\rangle
\]  
(206)

Hence:

\[
U |1, 1\rangle = \frac{i \sin \theta}{\sqrt{2}} [|2, 0\rangle + |0, 2\rangle] + \cos \theta |1, 1\rangle
\]  
(207)

which is, in general, an entangled state. Balanced beam-splitter \((\theta = \pi/2)\):

\[
U(\pi/2, 0) |1, 1\rangle = (|2, 0\rangle + |0, 2\rangle) / \sqrt{2}
\]  
(208)

Photon bunching due to their bosonic nature.
Field relaxation
Relaxation

Learn how to treat the coupling of a field mode to the external world. Examples of physical situations

- Propagation of a beam in a diffusive medium
- Field in a cavity with output coupling (laser)
- Field in a box with imperfect conductivity (real cavity)
Quantum system $\mathcal{S}$ (the field here) coupled to an environment $\mathcal{E}$ (other radiation modes through diffusion, electrons in the cavity walls...). Jointly in a pure state $|\Psi_{SE}\rangle$.

We are interested only in $\rho_{S}$, obtained by tracing the projector $|\Psi_{SE}\rangle \langle \Psi_{SE}|$ over the environment (the state of the environment is forever inaccessible).

We seek an evolution equation for $\rho_{S}$ alone.
Kraus operators

- Transformation of the system’s density matrix during a short time interval
  \[ \rho(t) \rightarrow \rho(t + \tau) \]  
- \( \tau \gg \tau_c \), correlation time of the reservoir observables, so that there are no coherent effects in the system-reservoir interaction
- This transformation is a ‘quantum map’
  \[ \mathcal{L}(\rho(t)) = \rho(T + \tau) \]
Kraus operators

Any physical quantum map (‘completely positive’) can be written as

\[ \mathcal{L}(\rho) = \sum_{\mu} M_{\mu} \rho M_{\mu}^\dagger \]  \hspace{1cm} (211)

with the normalization condition

\[ \sum_{\mu} M_{\mu}^\dagger M_{\mu} = \mathbb{1} \]  \hspace{1cm} (212)

There are at most \( N_S^2 \) ‘Kraus’ operators \( M_{\mu} \), which are not uniquely defined (same map when mixing the \( M_{\mu} \) by a linear unitary matrix \( V \): \( M'_{\mu} = V_{\mu \nu} M_{\nu} \)).
Lindblad equation

Kraus representation and differential representation of the map

\[ \rho(t + \tau) = \sum_{\mu} M_{\mu} \rho M_{\mu}^\dagger \approx \rho(t) + \frac{d\rho}{dt} \tau \] (213)

- Environment unaffected by the system: the \( M_{\mu} \)s do not depend upon time \( t \).
- They, however, depend clearly upon the tiny time interval \( \tau \).
- One and only one of the \( M_{\mu} \)s is thus of the order of unity and all others must then be of order \( \sqrt{\tau} \).

\[ M_0 = 1 - iK\tau \] (214)

\[ M_{\mu} = \sqrt{\tau}L_{\mu} \quad \text{for} \quad \mu \neq 0 \] (215)
**Lindblad equation**

\( K, \) having no particular properties, can be split in hermitian and anti-hermitian parts:

\[
K = \frac{H}{\hbar} - iJ ,
\]

(216)

where

\[
H = \frac{\hbar}{2} \left( K + K^\dagger \right)
\]

(217)

\[
J = \frac{i}{2} \left( K - K^\dagger \right)
\]

(218)

are both hermitian.

\[
M_0 = 1 - \frac{i \tau}{\hbar} H - J \tau
\]

(219)
Lindblad equation

Thus

\[ M_0 \rho M_0^\dagger = \rho - \frac{i \tau}{\hbar} [H, \rho] - \tau [J, \rho]_+ \]  

(220)

where \([J, \rho]_+ = J\rho + \rho J\) is an anti-commutator.

\[ M_0^\dagger M_0 = 1 - 2J\tau \]  

and thus, by normalization since \(\sum \mu \, M_\mu^\dagger M_\mu = 1\) (221)

\[ J = \frac{1}{2} \sum_{\mu \neq 0} L_\mu^\dagger L_\mu \]  

(222)

“Lindblad form” of the master equation

\[ \frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] + \sum_{\mu \neq 0} \left( L_\mu \rho L_\mu^\dagger - \frac{1}{2} L_\mu^\dagger L_\mu \rho - \frac{1}{2} \rho L_\mu^\dagger L_\mu \right) \]  

(223)
Quantum jumps

Consider a single time interval $\tau$ in the simple situation where the initial state is pure $\rho(0) = |\psi\rangle \langle \psi|$, with no Hamiltonian evolution. Then

$$\rho(\tau) = |\psi\rangle \langle \psi| + \tau \sum_{\mu} (L_{\mu} |\psi\rangle) \left( \langle \psi | L_{\mu}^{\dagger} \right)$$

(224)

- Density matrix at time $\tau$ is a statistical mixture of the initial pure state (with a large probability of order 1) and of projectors on the states $L_{\mu} |\psi\rangle$.
- The $L_{\mu}$s are ‘jump operators’ which describe a random (possibly large) evolution of the system which suddenly (at the time scale of the evolution) changes under the influence of the environment. Their form can often be guessed from a simple consideration of the relaxation mechanisms. E.g. a cavity at $T = 0$ K may loose a photon, but what else? Single jump operator proportional to $a$. 
Quantum jumps

- The quantum jump operators are not defined unambiguously. Again, the same master equation can be recovered from different sets of $M_{\mu}s$ (or $L_{\mu}s$) linked together by a unitary transformation matrix. Different choices correspond to the so-called ‘unravelings’ of the master equation.

- In some situations, the quantum jumps have a direct physical meaning. E.g. a field mode decaying in an ensemble of light detectors. The quantum jump then corresponds to a click of one detector. Different unravelings may then correspond to different ways of monitoring the environment, in this case to different types of detectors (photon counters, homodyne receivers...)

- In other situations, the quantum jumps are only an abstract representation of the system+environment evolution.
Quantum trajectories

- Even when the environment is not explicitly monitored, one may imagine that it is done. We then imagine we have full information about which quantum jump occurs when.

- The system is thus, at any time, in a pure state, which undergoes a stochastic trajectory in the Hilbert space, made up of continuous Hamiltonian evolutions interleaved with sudden quantum jumps.

- However, since we only imagine the information is available, we should describe the evolution of the density operator by averaging the system evolution over all possible trajectories.

- This leads to the quantum Monte Carlo approach, particularly interesting for numerical computations of relaxation. It provides also interesting physical insights into the relaxation process.
Quantum Monte Carlo algorithm

- Initialize the state (randomly chosen eigenstate $|\Psi\rangle$ of $\rho$)
- For each time interval $\tau$, evolve $|\Psi\rangle$ according to:
  - Compute $p_\mu = \tau \langle \Psi | L_\mu^\dagger L_\mu | \Psi \rangle$ and $p_0 = 1 - \sum_{\mu \neq 0} p_\mu$.
  - Use a (good) random number generator to decide upon the result of the measurement of $B$.
  - If the result of the measurement is zero, evolve $|\Psi\rangle$ with
    \[
    |\Psi\rangle \rightarrow \frac{1 - iH\tau/\hbar - J\tau}{\sqrt{p_0}} |\Psi\rangle \quad (225)
    \]
  - If the result of the measurement is $\mu \neq 0$, evolve $|\Psi\rangle$ by:
    \[
    |\Psi\rangle \rightarrow \frac{L_\mu}{\sqrt{\langle \Psi | L_\mu^\dagger L_\mu | \Psi \rangle}} |\Psi\rangle = \frac{L_\mu}{\sqrt{p_\mu/\tau}} |\Psi\rangle \quad (226)
    \]
- Repeat the procedure for many trajectories
- Average the projectors $\rho(t) = \frac{|\Psi(t)\rangle \langle \Psi(t)|}{\langle \Psi(t) | \Psi(t) \rangle}$
Quantum Monte Carlo algorithm

Interest of the Monte Carlo method:

- For each trajectory computes only a state vector with $N_S$ dimensions i.e. $N_S$ coupled differential equations, instead of $N_S^2$ equations for the full density operator.
- Needs a statistical sample of trajectories. A few hundreds is enough to get a qualitative solution. Method more efficient than the direct integration when $N_S$ is larger than a few hundreds.
- Gives a physical picture of the relaxation process.

An extremely useful method, with thousands of applications.
Jump operators for a relaxing field

Only two possible jump operators at finite temperature $T$

- $L_- = \sqrt{\kappa} a$: loss of a photon in the environment (even when $T = 0$)
- $L + - = \sqrt{\kappa} a^\dagger$: creation of a thermal excitation

Jump rates linked to the temperature of the environment

\begin{equation}
\kappa_+ = \kappa_- e^{-\hbar \omega / k_b T}
\end{equation}

Using

\begin{equation}
n_{th} = \frac{1}{e^{\hbar \omega / k_b T} - 1}
\end{equation}

we get

\begin{equation}
\frac{\kappa_-}{\kappa_+} = \frac{1 + n_{th}}{n_{th}}
\end{equation}

and write

\begin{equation}
\kappa_- = \kappa(1 + n_{th}) ; \quad \kappa_+ = \kappa n_{th}
\end{equation}
Lindblad equation

\[ \frac{d\rho}{dt} = -i\omega_c \left[ a^\dagger a, \rho \right] - \frac{\kappa(1 + n_{th})}{2} \left( a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger \right) \]
\[ - \frac{\kappa n_{th}}{2} \left( a a^\dagger \rho + \rho a a^\dagger - 2 a^\dagger \rho a \right) \]  

(231)

where we have discarded the vacuum energy. Note that all of the Hamiltonian part can be removed by an interaction representation (relaxation terms unchanged). For the photon number distribution:

\[ \frac{dp(n)}{dt} = \kappa(1 + n_{th})(n + 1)p(n + 1) + \kappa n_{th} np(n - 1) \]
\[ - [\kappa(1 + n_{th})n + \kappa n_{th}(n + 1)]p(n) \]  

(232)
 Thermal equilibrium

Detailed balance argument

\[ \kappa (1 + n_{th}) np(n) = \kappa n_{th} np(n - 1) \]  \hspace{1cm} (233)

leading to:

\[ \frac{p(n)}{p(n - 1)} = \frac{n_{th}}{1 + n_{th}} = e^{-\hbar \omega / k_b T} \]  \hspace{1cm} (234)

The expected Maxwell equilibrium
Relaxation of Fock states

At $T = 0$, relaxation of a Fock state

- **Jump**: removal of a photon
- **No jump**: non hermitian Hamiltonian

\[ H_e = -i\hbar J = -i\hbar \kappa a^\dagger a/2 \]  \hspace{1cm} (235)

Leaves photon number states invariant
Relaxation of Fock states

Relaxation of a 10-photon Fock state.
Relaxation of coherent states

Monte Carlo trajectory

- Jump: no evolution since $|\alpha\rangle$ is an eigenstate of $a$
- No jumps: evolution with non hermitian hamiltonian, equivalent to a complex mode frequency

$$|\beta\rangle \rightarrow |\beta e^{-\kappa \tau /2}\rangle$$

A coherent state remains coherent, with an exponentially damped amplitude.
Relaxation of coherent states

No change of the photon number in a quantum jump? A bayesian argument. $p(n|c)$ photon number distribution before the jump knowing that a jump occurs (‘click’ in the environment.) With

$$p(n, c) = p(c|n)p(n) = p(n|c)p_c \quad (237)$$

$$p(n|c) = p(n) \frac{p(c|n)}{p_c} = \frac{n}{\bar{n}} p(n) = e^{-\bar{n}} \frac{\bar{n}^{n-1}}{(n-1)!} = p(n-1) \quad (238)$$

A translated Poisson distribution with $\bar{n} + 1$ photons on the average. After jump photon number unchanged. Explains why the photon number distribution is invariant in a jump. Specific property of coherent states.
Chapter 2

Atom-field interaction
How to treat the quantum atom-field interaction?

Outline of the Chapter:
- Two-level atom
- Spontaneous emission
- Atomic relaxation
- Photodetection
- The dressed atom
Two-level atom
A two-level system

We consider the case of a radiation field mode nearly resonant on the transition between between the two levels $|g\rangle$ (lower, possibly ground level) and $|e\rangle$.

All other levels can be neglected. Boils down to the interaction of a field with a spin 1/2 system.
Atomic system

Two states $|e\rangle$ and $|g\rangle$ or $|+\rangle$ and $|-\rangle$ or $|0\rangle$ and $|1\rangle$ in quantum information science.

Operator basis set: Pauli operators

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(239)

\[
[\sigma_x, \sigma_y] = 2i\sigma_z
\]

(240)

Spin lowering and raising operators

\[
\sigma_+ = |+\rangle \langle -| = \frac{\sigma_x + i\sigma_y}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

(241)

\[
\sigma_- = |-\rangle \langle +| = \sigma_+^\dagger = \frac{\sigma_x - i\sigma_y}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

(242)

\[
[\sigma_z, \sigma_{\pm}] = \pm 2\sigma_{\pm}
\]

(243)
Atomic system

Most general observable \( \sigma_u \) with \( u = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \)

\[
\sigma_u = \begin{pmatrix}
\cos \theta & \sin \theta e^{-i\phi} \\
\sin \theta e^{i\phi} & -\cos \theta
\end{pmatrix}
\] (244)

Eigenvectors

\[
|+_u\rangle = |0_u\rangle = \cos \frac{\theta}{2} |+\rangle + \sin \frac{\theta}{2} e^{i\phi} |-\rangle
\] (245)

\[
|-_u\rangle = |1_u\rangle = -\sin \frac{\theta}{2} e^{-i\phi} |+\rangle + \cos \frac{\theta}{2} |-\rangle
\] (246)
Bloch sphere
Spin rotation

Rotation on the Bloch sphere by an angle $\theta$ around the axis defined by $\mathbf{v}$

$$R_v(\theta) = e^{-i(\theta/2)\sigma_v} = \cos \frac{\theta}{2} \mathbb{1} - i \sin \frac{\theta}{2} \sigma_v$$

(247)

e.g. angle $\theta$ around $\mathbf{u}_z$

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

(248)

with $R_z(\pi/2) |+_x\rangle = |+_y\rangle$ and $R_v(2\pi) = -\mathbb{1}$
Atom-field Hamiltonian

- Free atomic Hamiltonian:
  \[ H_0 = \frac{\hbar \omega_{eg}}{2} \sigma_z \]  
  Generates a rotation of the Bloch vector (Larmor precession) at angular frequency \( \omega_{eg} \) around \( Oz \).

- Electric dipole operator:
  \[ D = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix} = d \sigma_x = d(\sigma_+ + \sigma_-) = d\epsilon_d \langle g | e \rangle + \text{h.c.} \]  
  where \( \epsilon_d \) describes the polarization and amplitude of the atomic transition. A priori complex (circularly polarized transitions) but taken here as real for clarity.
Atom-field Hamiltonian

With the hypotheses:

- Weak field (small in atomic units)
- Dipole approximation: the atomic size is much smaller than the mode wavelength

the atom-field interaction Hamiltonian reads

$$ H_i = - \mathbf{D} \cdot \mathbf{E}(0) $$

Electric field in the plane mode basis

$$ \mathbf{E}(0) = i \sum_{\ell} \sqrt{\frac{\hbar \omega_\ell}{2 \epsilon_0 N}} a_\ell \epsilon_\ell + \text{h.c.} $$

For nearly resonant modes (dominant effect in general), two of the four terms in $\mathbf{D} \cdot \mathbf{E}(0)$ can be neglected (RWA approximation)
Spontaneous emission in free space
Fermi Golden rule

Decay of an excited atom in the continuum of free-space modes. Initial state $|e, 0\rangle$. Continuum of final states $|g, 1_\ell\rangle$. Use the Fermi Golden rule to compute separately the rate of photon emission in all directions:

$$
\Gamma = \int d\Gamma d\Omega \quad (253)
$$

$$
d\Gamma = \sum_{\epsilon_\ell} \frac{2\pi}{\hbar} |W|^2 d\rho(E = \hbar \omega_0, d\Omega) \quad (254)
$$

Density of states $d\rho = \rho d\Omega/4\pi$ where

$$
\rho(\nu) = \frac{8\pi}{2c^3} \nu \nu^2 d\nu \quad (255)
$$

With $\rho(E)dE = \rho(\nu)d\nu$ for $E = h\nu = \hbar \omega_0$

$$
\rho(E) = \frac{\nu}{2\pi^2 c^3} \frac{1}{\hbar} \left(\frac{E}{\hbar}\right)^2 \quad (256)
$$
Chapter 2: Atom-field interaction

Fermi Golden rule

Finally

\[ d\rho(E = \hbar\omega_0, d\Omega) = \frac{\mathcal{V}}{8\pi^3} \frac{\omega_0^2}{\hbar c^3} d\Omega \]  

Coupling

\[ |W|^2 = |\langle g, 1_\ell | D \cdot E | e, 0 \rangle|^2 \]  

Without loss of generality

\[ \epsilon_d = u_z \]  

\[ |W|^2 = \left| d u_z \cdot \epsilon^*_\ell \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0 \mathcal{V}}} \right|^2 \]  

We can now evaluate the rate

\[ d\Gamma = \sum_{\epsilon_\ell} \frac{1}{8\pi^2 \epsilon_0 c^3} \frac{\omega_0^3}{\hbar} \frac{|d|^2}{|u_z \cdot \epsilon^*_\ell|^2} d\Omega \]
Fermi Golden rule

Expand $u_z$ on the basis of $u_k$ (propagation direction) and two orthogonal linear polarizations $\epsilon_1$ and $\epsilon_2$:

$$(u_z \cdot \epsilon_1^*)^2 + (u_z \cdot \epsilon_2^*)^2 = 1 - (u_z \cdot u_k)^2 = 1 - \cos^2 \theta = \sin^2 \theta$$

(262)

Integration over solid angle:

$$\Gamma = \frac{1}{8\pi^2\epsilon_0} \frac{\omega_0^3 |d|^2}{\hbar} \int_0^{2\pi} \int_0^\pi \sin^3 \theta \, d\theta \, d\phi$$

(263)

and finally

$$\Gamma = \frac{\omega_0^3 |d|^2}{3\pi\omega_0 \hbar c^3}$$

(264)

An essential result in atomic physics.
Wigner-Weisskopf

A more detailed insight. Compute the complete evolution of the atom-field state. Predicts two effects:

- A spontaneous emission rate $\Gamma$ (the same as by Fermi Golden rule)
- A level shift (‘Lamb shift’) $\Delta$

A central problem in QED

$\Delta$ diverges

Only resolved with the renormalization techniques of quantum field theory (2017 is the 70th anniversary of the Shelter Island congress). We need not take care of that. Only use the experimentally determined atomic frequency, which incorporates the renormalized shift.
Atomic relaxation
Jump operator approach to atomic relaxation

Zero temperature model. A single jump operator (describing photon emission in a downwards transition)

\[ L = \sqrt{\Gamma} \sigma_- \tag{265} \]

with \( \Gamma = 1/ T_1 \) (‘longitudinal relaxation time’). Lindblad equation

\[ \frac{d\rho}{dt} = \Gamma \left( \sigma_- \rho \sigma_+ - \frac{1}{2} \sigma_+ \sigma_- \rho - \frac{1}{2} \rho \sigma_+ \sigma_- \right) \tag{266} \]

We leave as an exercise the Lindblad equation at a finite temperature.
Lindblad equation

With

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix}$$  \hspace{1cm} (267)$$

the solution of the Lindblad equation is

$$\frac{d\rho_{ee}}{dt} = -\Gamma \rho_{ee}$$  \hspace{1cm} (268)$$

$$\frac{d\rho_{eg}}{dt} = -\frac{\Gamma}{2} \rho_{eg}$$  \hspace{1cm} (269)$$

- Relaxation of excited state population with a rate $\Gamma$.
- Relaxation of coherence with a rate $\Gamma/2$ (compatible with $\rho_{eg} \leq \sqrt{\rho_{ee}\rho_{gg}}$)
Phase damping

Model atomic relaxation due to random fields altering the atomic frequency and scrambling the coherence phase.

- Jump operator $\sqrt{\gamma/2}\sigma_z$ with $\gamma = 1/T_2$ the ‘transverse’ relaxation rate and $T_2$ the transverse relaxation time. Models sudden phase shifts of coherences.
- No damping of the populations, but coherences damped at rate $\gamma$.
- Complete Lindblad equation with spontaneous emission

$$
\frac{d\rho_{ee}}{dt} = -\Gamma \rho_{ee}
$$

(270)

$$
\frac{d\rho_{eg}}{dt} = -\frac{\Gamma}{2} \rho_{eg} - \gamma \rho_{eg} = -\gamma' \rho_{eg}
$$

(271)

where we define the total relaxation rate of the coherence by:

$$
\gamma' = \gamma + \frac{\Gamma}{2}
$$

(272)
Relaxation of a state superposition

Case of an initial superposition state \(|\Psi_0\rangle = (1/\sqrt{2})(|e\rangle + |g\rangle)\). Analysis in terms of the Monte Carlo trajectories.

- No jump evolution. With \(|\Psi(t)\rangle = c_e |e\rangle + c_g |g\rangle\) and use effective Hamiltonian

\[
H = -i\hbar J = -\frac{i\hbar}{2} \Gamma \sigma_+ \sigma_- = -\frac{i\hbar}{2} \Gamma |e\rangle \langle e| \tag{273}
\]

\[
i\hbar \frac{dc_e}{dt} = -\frac{i\hbar}{2} \Gamma c_e \quad c_e(t) = c_e(0)e^{-\Gamma t/2} \quad \frac{dc_g}{dt} = 0 \tag{274}
\]

\[
|\Psi(t)\rangle = \frac{1}{|c_e(0)|^2 e^{-\Gamma T} + |c_g(0)|^2} \left( c_e(0)e^{-\Gamma t/2} |e\rangle + c_g(0) |g\rangle \right) \tag{275}
\]

A negative detection (no photon emitted) changes the system’s state.

- Jump: state becomes \(|g\rangle\). No further evolution.
Photodetection
Photodetector model

A simple single system photodetector. A ground state $|g\rangle$ and a continuum of excited states $|e_i\rangle$. Transition to excited state is a click.

Detector Hamiltonian

$$H_d = \sum_i \hbar \omega_i |e_i\rangle \langle e_i|$$

(276)

Detector-field interaction $-\mathbf{D} \cdot \mathbf{E}$ with

$$\mathbf{D} = \sum_i d_i (\epsilon_i |g\rangle \langle e_i| + \epsilon_i^* |e_i\rangle \langle g|)$$

(277)

Hence, within irrelevant factors

$$H_i = \sum_i \kappa_i |e_i\rangle \langle g| E^+ + \text{h.c.}$$

(278)
Photodetector model

Interaction representation $a_\ell \rightarrow a_\ell \exp(-i\omega_\ell t)$, $|e_i\rangle\langle g| \rightarrow \exp(i\omega_i t)|e_i\rangle\langle g|$

$$\tilde{H}_i = \sum_{i} \kappa_i e^{i\omega_i t} |e_i\rangle\langle g| E^+(t) + \text{h.c.} \quad (279)$$

Initial condition

$$|\Psi(0)\rangle = |g\rangle \otimes |\Psi_f\rangle \quad (280)$$

State at time $t$

$$|\Psi(t)\rangle = |g\rangle \otimes |\Psi_f\rangle + \frac{1}{i\hbar} \int_{0}^{t} \tilde{H}_i(t') |\Psi(t')\rangle \, dt' \quad (281)$$

First-order perturbative solution by replacing in the r.h.s. $|\Psi(t')\rangle$ by $|\Psi(0)\rangle = |g\rangle \otimes |\Psi_f\rangle$. 

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Chapter 2: Atom-field interaction

Photodetection

Photodetector model

Interaction representation $a_\ell \rightarrow a_\ell \exp(-i\omega_\ell t)$, $|e_i\rangle\langle g| \rightarrow \exp(i\omega_i t)|e_i\rangle\langle g|$

$$\tilde{H}_i = \sum_{i} \kappa_i e^{i\omega_i t} |e_i\rangle\langle g| E^+(t) + \text{h.c.} \quad (279)$$

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First-order perturbative solution by replacing in the r.h.s. $|\Psi(t')\rangle$ by $|\Psi(0)\rangle = |g\rangle \otimes |\Psi_f\rangle$. 

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J.M. Raimond

Photons and Atoms

January 10, 2017

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Photodetector model

Noting that, in $\tilde{H}_i$, $|g\rangle \langle e_i| E^-$ gives zero on the initial state

$$|\Psi(t)\rangle = |g\rangle \otimes |\Psi_f\rangle + \frac{1}{i\hbar} \sum_i \kappa_i \left[ \int_0^t dt' e^{i\omega_i t'} E^+(t') |\Psi_f\rangle \right] \otimes |e_i\rangle$$

(282)

Probability for having a count at time $t$

$$p_e = \sum_i |\langle e_i | \Psi \rangle|^2 = \sum_i \langle \Psi | e_i \rangle \langle e_i | \Psi \rangle$$

(283)

$$p_e = \frac{1}{\hbar^2} \sum_i |\kappa_i|^2 \int_0^t dt' \int_0^{t'} dt'' e^{i\omega_i (t'-t'')} \langle \Psi_f | E^-(t'') E^+(t') |\Psi_f\rangle$$

(284)
Photodetection current

For a high density of final states

\[ \sum_i \rightarrow \int d\omega \rho(\omega); \quad \int d\omega e^{i\omega(t'-t'')} = \pi \delta(t' - t'') \]  \hspace{1cm} (285)

Hence

\[ p_e(t) \propto \int_{t_0}^{t} dt' \langle \Psi_f | E^-(t')E^+(t') | \Psi_f \rangle \]  \hspace{1cm} (286)

With a large set of photo-detecting systems the ‘photocurrent’ is proportional to

\[ I(t) = \langle \Psi_f | E^-(t)E^+(t) | \Psi_f \rangle \]  \hspace{1cm} (287)

An essential result in quantum optics. Can be generalized to more complex field correlations eg

\[ G_2(r_1, r_2, t, \tau) = l_1(r_1, t)l_2(r_2, t + \tau) = \langle \Psi_f | \hat{G}_2 | \Psi_f \rangle \]  \hspace{1cm} (288)

where

\[ \hat{G}_2 = E^-(r_1, t)E^-(r_2, t + \tau)E^+(r_2, t + \tau)E^+(r_1, t) \]  \hspace{1cm} (289)
The dressed atom
The dressed atom model

A two level atom coupled to a single mode of the radiation field. Coherent coupling larger than dissipative process.

- An atom in an intense laser field
- Cavity quantum electrodynamics

Fruitful to treat atom and mode as a single quantum system. Spontaneous emission and shifts can be added later as a small perturbation.
Chapter 2: Atom-field interaction

The dressed atom

Hamiltonian

\[ H = H_a + H'_c + H_{ac} \]  \hspace{1cm} (290)

where \( H_a \) and \( H'_c = \hbar \omega_c N \) are the atom and field Hamiltonians. In the RWA

\[ H_{ac} = -i\hbar \frac{\Omega_0}{2} \left[ a\sigma_+ - a^\dagger \sigma_- \right] \]  \hspace{1cm} (291)

where we introduce the ‘vacuum Rabi frequency’ \( \Omega_0 \) (assumed to be real):

\[ \Omega_0 = 2 \frac{d\mathcal{E}_0 \epsilon_d^* \cdot \epsilon_c}{\hbar} \]  \hspace{1cm} (292)

Atom-field detuning

\[ \Delta_c = \omega_{eg} - \omega_c \]  \hspace{1cm} (293)
Uncoupled states
Eigenenergies and eigenvectors

In the $n$th doublet

$$H_n = \hbar \omega_c (n + 1/2) \mathbb{1} + V_n$$  \hspace{1cm} (294)$$

with:

$$V_n = \frac{\hbar}{2} \begin{pmatrix} \Delta_c & -i \Omega_n \\ i \Omega_n & -\Delta_c \end{pmatrix} = \frac{\hbar}{2} [\Delta_c \sigma_Z + \Omega_n \sigma_Y]$$ \hspace{1cm} (295)$$

and

$$\Omega_n = \Omega_0 \sqrt{n + 1}$$ \hspace{1cm} (296)$$
Eigenenergies and eigenvectors

Eigenvalues:

\[ E_n^\pm = (n + 1/2) \frac{\hbar}{2} \omega_c \pm \frac{\hbar}{2} \sqrt{\Delta_c^2 + \Omega_n^2} \]  

(297)

with

\[ \tan \theta_n = \frac{\Omega_n}{\Delta_c} \]  

(298)

Eigenvectors

\[ |+, n\rangle = \cos(\theta_n/2) |e, n\rangle + i \sin(\theta_n/2) |g, n + 1\rangle \]

\[ |-, n\rangle = \sin(\theta_n/2) |e, n\rangle - i \cos(\theta_n/2) |g, n + 1\rangle \]  

(299)

The ‘dressed states’.
Dressed states

\[ |+,n\rangle \]
\[ |-,n\rangle \]
\[ |e,n\rangle \]
\[ |g,n+1\rangle \]

Energy vs. \( \Delta_c/\Omega_n \)
Chapter 2: Atom-field interaction

The dressed atom

Resonant case: Rabi oscillation

\( \theta_n = \pi/2 \) for all \( n \) values.

\[ | \pm, n \rangle = \left[ | e, n \rangle \pm i | g, n + 1 \rangle \right] / \sqrt{2} \]  \hspace{1cm} (300)

Initial state \( | \psi_e(0) \rangle = | e, n \rangle \)

\[ | \psi_e(0) \rangle = \left[ | +, n \rangle + | -, n \rangle \right] / \sqrt{2} \]  \hspace{1cm} (301)

Interaction representation with respect to the constant \( \hbar \omega_c (n + 1/2) \mathbf{1} \). At time \( t \)

\[ \left| \tilde{\psi}_e(t) \right\rangle = \left[ | +, n \rangle e^{-i\Omega_n t/2} + | -, n \rangle e^{i\Omega_n t/2} \right] / \sqrt{2} \]  \hspace{1cm} (302)

In the uncoupled basis

\[ \left| \tilde{\psi}_e(t) \right\rangle = \cos \frac{\Omega_n t}{2} | e, n \rangle + \sin \frac{\Omega_n t}{2} | g, n + 1 \rangle \]  \hspace{1cm} (303)

For an atom initially in \( g \)

\[ \left| \tilde{\psi}_g(t) \right\rangle = - \sin \frac{\Omega_n t}{2} | e, n \rangle + \cos \frac{\Omega_n t}{2} | g, n + 1 \rangle \]  \hspace{1cm} (304)
Non-resonant (dispersive) coupling

Large detuning case (dispersive regime) \(|\Delta_c| \gg \Omega_n\)

\[
\begin{align*}
|+,n\rangle & \quad |e,n\rangle \\
|g,n+1\rangle & \quad |-,n\rangle \\
|+,n-1\rangle & \quad |e,n-1\rangle \\
|g,n\rangle & \quad |-,n-1\rangle
\end{align*}
\]

\[
\begin{align*}
\omega_{eg} + \delta\omega_{eg} & \quad \omega_c + \delta\omega_c \\
\omega_c + \delta\omega_c & \quad \omega_c + \delta\omega_c
\end{align*}
\]
Non-resonant (dispersive) coupling

\[ E_n^\pm = (n + 1/2) \hbar \omega_c \pm \hbar \left( \frac{\Delta_c}{2} + \frac{\Omega_n^2}{4\Delta_c} \right) \]  \hspace{1cm} (305)

\[ \Delta_{e,n} = \hbar(n + 1)s_0 \; ; \; \Delta_{g,n} = -\hbar ns_0 \]  \hspace{1cm} (306)

with:

\[ s_0 = \frac{\Omega_0^2}{4\Delta_c} \]  \hspace{1cm} (307)

Two complementary effects

- Atomic frequency change (light shifts and Lamb shift)
  \[ \delta \omega_{eg} = (2n + 1)s_0 \]  \hspace{1cm} (308)

- Atomic state-dependent mode shift (index of refraction effect)
  \[ \delta_{e\omega_c} = s_0 \]  \hspace{1cm} (309)
  \[ \delta_{g\omega_c} = -\delta_{e\omega_c} = -s_0 \]  \hspace{1cm} (310)
Autler Townes splitting

The large field version of the vacuum Rabi splitting. Atom driven by an intense laser field resonant on the $e/g$ transition. Two dressed states with a splitting $\Omega \sqrt{n}$ nearly independent of photon number $n$ for a large coherent field. Dressed states are superpositions of $e$ and $g$ with equal weights. Probe the system on the $h$ to $g$ transition where $h$ is a third level.

- For a negligible laser intensity: a single line.
- For a strong laser: two lines corresponding to the excitation of the two dressed levels, separated by the Rabi splitting.
Mollow triplet

Fluorescence of the dressed levels in a strong resonant laser field. Rabi splitting nearly the same for all relevant photon numbers. Add atomic relaxation by spontaneous emission. Emission possible on all transitions between the levels.

A triplet of lines

- Atomic frequency
- Sidebands at the Rabi frequencies
Chapter 3

Cavity quantum electrodynamics
A paradigmatic situation: The coherent coupling of a single two-level atom to a single field mode overwhelms all other dissipative processes. Outline of the Chapter:

- Experimental tools
- Resonant CQED
- Dispersive CQED
- Perspectives

Switch to a power point presentation...