QCD Sum Rules Approach for the Gluon Condensate and Deconfinement

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This talk is based on the following article:

- Quark deconfinement and Gluon Condensate in a weak magnetic field from QCDSum Rules.

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There are two (at least two) phase transitions that may occur in QCD at finite temperature and/or density:

1) Deconfinement due to color screening
2) Chiral symmetry restoration: Moving from a Nambu-Goldstone to a Wigner-Weyl realization

Which are the relevant order parameters in each case?

Both transitions seem to occur approximately at the same temperature
General Aspects:

An order parameter is a quantity that vanishes in a certain phase, being finite in a second one.

The relevant physical variables are temperature (T) and baryon chemical potential ($\mu_B$).

Normally the Polyakov loop (confinement) and the quark condensate (chiral symmetry restoration) are used as order parameters.

When $\mu_B = 0$ and $T \neq 0$ lattice results provide a consistent picture, resulting in a similar $T_c$ for both transitions in the range $170 \text{ MeV} < T_c < 200 \text{ MeV}$ (finite quark masses).

However.....
For finite Baryon Chemical Potential, the fermion determinant becomes complex and lattice simulations are not possible.

So, perhaps we need a new variable, instead of the Polyakov Loop, for discussing deconfinement.

An attractive possibility: the continuum threshold of the hadronic resonance spectral function. Phenomenological order parameter. This discussion can be done in the frame of the extended QCD Sum Rules ($\mu_B \neq 0$ and $T \neq 0$) program.
In fact, in a series of papers we have shown that, as function of temperature, the continuum threshold starts moving to the left and, at the same time the width of the resonances become wider. In fact the width diverges at the critical temperature. 

This means that the resonances melt, and the critical temperature in this picture occurs when the continuum threshold goes into the threshold of the channel.

We found, however, that charmonium and bottomonium states behave differently!!
Realistic Spectral Function

\[
\text{Im} \quad \Pi \equiv E^2
\]
Realistic Spectral Function (T)

\[ \text{Im } \Pi \equiv E^2 S_0(T) \]
Here, however, I want to report on a work we have done exploring the influence of magnetic fields on the continuum threshold and on the Gluon Condensate in the frame of QCD Sum Rules (more precisely using FESR). So, no temperature Effects in this presentation.

We are now developing the extension of this work to a finite temperature scenario (including also density effects). So: Work in progress.

So, let me start by remembering the basic concepts and ideas behind the QCD Sum Rules approach to hadron physics.
For this purpose we will use QCD Sum Rules.

OPERATOR PRODUCT EXPANSION OF CURRENT CORRELATORS AT SHORT DISTANCES
(BEYOND PERTURBATION THEORY)

CAUCHY’S THEOREM IN THE COMPLEX ENERGY (SQUARED) S-PLANE
CONFINEMENT

• STRONG MODIFICATION TO QUARK & GLUON PROPAGATORS NEAR THE MASS SHELL

• INCORPORATE CONFINEMENT THROUGH A PARAMETRIZATION OF PROPAGATOR CORRECTIONS IN TERMS OF QUARK & GLUON VACUUM CONDENSATES
\[ S_F = \frac{i}{p - m} \quad \Rightarrow \quad \frac{i}{p - m + \Sigma(p^2)} \]

\[ D_F = \frac{i}{k^2} \quad \Rightarrow \quad \frac{i}{k^2 + \Lambda(p^2)} \]
QUARK CONDENSATE

\[ \langle 0 | \bar{q} q | 0 \rangle \]
GLUON CONDENSATE

\[ \langle 0 | \alpha_s G^a_{\mu \nu} G^a_{\mu \nu} | 0 \rangle \]
Let us consider the two point charged axial-vector current correlator

\[ \Pi_{\mu\nu}(q^2) = i \int d^4x e^{iqx} \langle 0 | T(A_\mu(x), A_\nu^\dagger(0)) | 0 \rangle \]

\[ = (-g_{\mu\nu}q^2 + q_\mu q_\nu) \Pi_A(q^2) + q_\mu q_\nu \Pi_0(q^2) \]

with

\[ A_\mu(x) =: \bar{d}(x) \gamma_\mu \gamma_5 u(x) : \]
Invoking the OPE including non-perturbative corrections due to the condensates

\[ \Pi_0(Q^2)|_{QCD} = C_0 \hat{I} + \sum_{N=1} \frac{C_{2N}(Q^2, \mu^2)}{Q^{2N}} \langle \hat{O}_{2N}(\mu^2) \rangle \]

with

\[ Q^2 \equiv -q^2, \quad \langle \hat{O}_{2N}(\mu^2) \rangle \equiv \langle 0|\hat{O}_{2N}(\mu^2)|0 \rangle \]

The first term stands for the purely perturbative contribution
Ignoring radiative corrections

\[ C_0 \hat{I} = \frac{1}{4\pi} \ln \left( \frac{-s}{\mu^2} \right) \left[ 1 + \mathcal{O}(\alpha_s(s)) \right] \]

In the chiral limit the dimension d=4 is proportional to the Gluon Condensate

\[ C_4 \langle \hat{O}_4 \rangle = \frac{\pi}{3} \langle \alpha_s G^2 \rangle \]
The second pillar behind the QCD Sum Rules has to do with analytic considerations (Cauchy Theorem)

The famous Packman contour
Since there are no poles inside the contour, for any integer $N$ we have

\[
- \frac{1}{2\pi i} \oint_{C(|s_0|)} dss^{N-1} \Pi_0^{\text{QCD}}(s) = \frac{1}{\pi} \int_0^{s_0} dss^{N-1} \text{Im} \Pi_0^{\text{HAD}}(s)
\]

In fact, for an arbitrary analytic function $f(s)$

\[
\frac{1}{\pi} \int_0^{s_0} ds f(s) \text{Im} \Pi_0(s) = -\frac{1}{2\pi i} \oint_{C(|s_0|)} ds f(s) \Pi_0(s)
\]
By integrating along the contour, Cauchy’s theorem produces immediately a set of FESR. On the cut we have the hadronic resonances. On the circle a description in terms of QCD degrees of freedom is assumed to be valid.

\[ \int_{C(s)} f(s) \, s^N \, ds = 0 \]

Since the Hadronic spectral function has the form

\[ \text{Im} \Pi(s)_{\text{HAD}} = \text{Im} \Pi(s)_{\text{POLE}} + \text{Im} \Pi(s)_{\text{RES}} \theta(s_0 - s) + \text{Im} \Pi(s)_{\text{PQCD}} \theta(s - s_0) \]
We immediately find

\[ \int_{s_0} f(s) \left\langle HAD \right| \Pi(s) \right|_{HAD} = -\frac{1}{2\pi i} \int_{C\left(|s_0|\right)} ds f(s) \left\langle QC\right| \Pi(s) \right|_{QC\right)} \]

And this provides us with the following set of Finite Energy Sum Rules

\[ (-)^N C_{2N+2} \left\langle 0 | \hat{O}_{2N+2} | 0 \right\rangle = \int_{0}^{s_0} ds s^N \frac{1}{\pi} Im \Pi(s) \left|_{HAD} = s_0^{N+1} M_{2N+2}(s_0) \right) \]
Where the dimensionless perturbative PQCD moments $M_N$ are given by

$$M_{2N+2}(s_0) = \frac{1}{s_0^{N+1}} \int_0^{s_0} ds \ s^N \ \frac{1}{\pi} \ \text{Im} \ \Pi(s)|_{PQCD}$$

If PQCD reduces only to the quark loop we have (axial-axial correlator)

$$\int_{\mathcal{C}(s_0)} ds \ s^{N-1} \ \ln\left(\frac{S}{\mu^2}\right) = \frac{i}{2\pi} \ \frac{S_0^N}{N}$$

$$\int_{\mathcal{C}(s_0)} ds \ \frac{C_4 \ < O_4 >}{S^2} = 2\pi i \ \frac{C_4 \ < O_4 >}{4\pi^2} \ \delta_{N2}$$
In fact, this result gives us a one to one correspondence between condensates and momenta.

\[
(-)^{N-1} C_{2N} \langle \hat{O}_{2N} \rangle = 4 \pi^2 \int_{0}^{s_0} ds s^{N-1} \frac{1}{\pi} \text{Im} \Pi_0(s) - \frac{s_0^N}{N} [1 + \mathcal{O}(\alpha_s)] (N = 1, 2, \cdots)
\]

FESR (Finite Energy Sum Rules)
Huge Magnetic fields are produced in peripheral heavy-ion collisions
Peripheral Heavy ion collisions
Time evolution of a uniform magnetic field in a heavy-ion collision

- Very intense field at early collision times

D. E. Kharzeev, L. D. McLerran, H. J. Warringa,
Time evolution of a constant B field in a heavy-ion collision

- Magnetic field rapidly decreasing function of collision time

When a magnetic field is present the perturbative coefficient as well as the Wilson coefficients associate to the condensates should be replaced (because the magnetic field carries dimension of energy square)

\[
C_0 \ln \left( \frac{-s}{\mu^2} \right) \rightarrow C_0 \ln \left( \frac{-s}{\mu^2} \right) + \sum_{n=1} C_0^{(n)} \frac{(eB)^n}{s^n}
\]

\[
C_{2N} \rightarrow \sum_{m=0}^{C_{2N}} C_{2N}^{(m)} \frac{(eB)^m}{s^m}
\]

Where the coefficients are dimensionless quantities that can be calculated at a certain order in eB
In this way we get new FESR with magnetic field

\[- \sum_{m=0}^{N-1} (-1)^{N-m} C_{2(N-m)}^{(m)} (eB)^m \langle O_{2(N-m)} \rangle \]

\[= \frac{1}{\pi} \int_0^{s_0} dss^{N-1} \text{Im} \Pi_0^{\text{HAD}}(s) - \frac{C_0}{N} s_0^N + C_0^{(N)} (eB)^N \]

Notice that the presence of the magnetic field mixes operators of different dimensions in the FESR. For N=1, 2 we get

\[0 = \frac{1}{\pi} \int_0^{s_0} ds \text{Im} \Pi_0^{\text{HAD}}(s) - C_0 s_0 + C_0^{(1)} (eB), \]

\[-C_4^{(0)} \langle O_4 \rangle + C_2^{(1)} (eB) \langle O_2 \rangle = \frac{1}{\pi} \int_0^{s_0} dss \text{Im} \Pi_0^{\text{HAD}}(s) - \frac{C_0}{2} s_0^2 + C_0^{(2)} (eB)^2. \]
\[ \Pi(q^2) = i \int d^4 x e^{i q x} \langle 0 | T(J(x) J^+(0)) | 0 \rangle \]

\[ J(x) \Rightarrow A_\mu(x) \big|_j \Rightarrow f_\pi \]

\[ \langle 0 | A_\mu(0) | \pi(q) \rangle = f_\pi \ q_\mu \]
In the hadronic sector we calculate the same correlator, using an interpolating current such that

\[
\langle 0| A_\mu(0) |\pi(p)\rangle = i f_\pi p_\mu
\]

\[
-i \partial^\mu \langle 0| A_\mu(0) |\pi(p)\rangle = f_\pi M_\pi^2
\]

In the presence of a magnetic field we will have

\[
A_\mu = -f_\pi D_\mu \pi^+ = -f_\pi (\partial_\mu - ie A_\mu) \pi^+
\]

We used the gauge where

\[
A_\mu = (B/2)(0, -y, x, 0)
\]
In this way we get

\[ \Pi^{\text{HAD}}_{\mu\nu}(x, y) \equiv \langle 0 | T(A_{\mu}(x), A^\dagger_{\nu}(y)) | 0 \rangle \]

\[ = i f_\pi^2 \langle 0 | T[D_{\mu}\pi^+(x)D^*_\nu\pi^-(y)] | 0 \rangle \]

\[ = i f_\pi^2 D_{\mu}(x)D^*_\nu(y)G_\pi(x, y) \]

\[ = ie^{ie\Phi(x,y)} f_\pi^2 (\partial/\partial x^\mu)(\partial/\partial y^\nu) \tilde{G}_\pi(x - y) \]

\[ \equiv e^{ie\Phi(x,y)} \Pi^{\text{HAD}}_{\mu\nu}(x - y), \]

The phase factor

\[ \Phi(x, y) = \int_x^y A(\xi)d\xi. \]

can be gauged away
We only keep the invariant part the translational invariant part. In the Fourier space we have

\[ \Pi_0^{\text{HAD}}(q^2) = if_\pi^2 \tilde{G}_\pi(q^2). \]

And we have Schwinger's proper time representation

\[ \tilde{G}_\pi(q^2) = \int_0^{\infty} \frac{d\tau}{\cos(eB\tau)} e^{i[\frac{q^2}{2} - \frac{q^2}{eB}\tan(eB\tau)/eB\tau + ie]} \]

Notice that we are in the chiral limit, therefore \( m_\pi = 0 \)
We have used the following notation

\[
\begin{align*}
g_{\mu\nu} &= g_{\mu\nu}^{\parallel} - g_{\mu\nu}^{\perp} \\
g_{\mu\nu}^{\parallel} &= \text{diag}(1, 0, 0, -1) \\
g_{\mu\nu}^{\perp} &= \text{diag}(0, 1, 1, 0).
\end{align*}
\]

Which implies

\[
\begin{align*}
a \cdot b &= (a \cdot b)^{\parallel} - (a \cdot b)^{\perp} \\
(a \cdot b)^{\parallel} &= g_{\mu\nu}^{\parallel} a^\mu b^\nu \\
&= a_0 b_0 - a_3 b_3 \\
(a \cdot b)^{\perp} &= g_{\mu\nu}^{\perp} a^\mu b^\nu \\
&= a_1 b_1 + a_2 b_2 \\
g_{\mu\nu} g^{\mu\nu} &= 2 \\
g_{\mu\nu}^{\perp} g^{\mu\nu} &= -2.
\end{align*}
\]
The charged pion propagator can be expressed as a sum over Landau levels

\[ \tilde{G}_\pi(q^2) = 2i \sum_{l=0}^{\infty} \frac{(-1)^l L_l(2q_\perp^2 / eB) e^{-q_\perp^2 / eB}}{q_\parallel^2 - (2l + 1)eB} \]

Here Lorentz invariance is lost. We will choose the static frame where the space part of the four momentua vanishes. Since, however, the magnetic field separates things into longitudinal and transverse parts, \( q_0 \) always appears in combination with \( q_3 \). We will ignore the transverse parts, and at the end we will allow \( q_3 \to 0 \) (if it is necessary)
Therefore, we have

\[ \tilde{G}_\pi(q^2) = 2i \sum_{l=0}^{\infty} \frac{(-1)^l}{q_\parallel^2 - (2l + 1)eB} \]

And the spectral function in the hadronic sector looks like

\[ \Pi_0^{\text{HAD}}(q_\parallel^2 = s) = -2f_\pi^2 \sum_{l=0}^{\infty} \frac{(-1)^l}{s - (2l + 1)eB} \]

It can be shown that

\[ \text{Im}\Pi_0^{\text{HAD}}(s) = f_\pi^2 \pi \delta(s - eB). \]

provided

\[ eB < s_0 < 3eB \]
In this way we have

\[
\frac{1}{\pi} \int_0^{s_0} ds s^{N-1} \text{Im} \Pi_0^{\text{HAD}}(s) = f_\pi^2 (eB)^{N-1}
\]

The FESR with \( N=1 \) and \( N=2 \) imply

\[
0 = f_\pi^2 - C_0 s_0 + C_0^{(1)} (eB) \\
-C_4 \langle O_4 \rangle = f_\pi^2 (eB) - \frac{C_0}{2} s_0^2 + C_0^{(2)} (eB)^2
\]

We need to calculate perturbatively the coefficients \( C_0^{(1)} \) and \( C_0^{(2)} \).
For this we use the weak field expansion of the fermionic propagators up to the order $O(B^2)$

\[
iS_B(k) = i \frac{k}{k^2} - (e_q B) \frac{\gamma_1 \gamma_2 (\gamma \cdot k)_\parallel}{k^4} \left[ \frac{k^2 (\gamma \cdot k)_\parallel - k^2 (\gamma \cdot k)_\perp}{k^8} \right]
\]

Notice that the field is weak with respect to $S_0$. The pQCD contribution to the correlator is
The propagators inside the loop are full magnetic field dependent propagators. So we proceed in a systematic way using the weak expansion for the propagators.

This first order corrections vanish.

The first non-vanishing contributions are of second order in powers of eB.
Second order corrections to the correlator

\[ \Pi_{\mu\nu}^{(11)}(q^2) = -iN_c(q_u q_d B^2) \]
\[ \times \int \frac{d^4k}{(2\pi)^4} \frac{\text{Tr}[\gamma_\mu \gamma_1 \gamma_2 [\gamma \cdot (k - q) || \gamma \cdot k ||]}{(k - q)^4 k^4} \]
If we define

\[ \tilde{\Pi}_0^{(11)}(q^2) \equiv q^\mu q^\nu \Pi^{(11)}_{\mu\nu}(q^2), \]

(a projection on the longitudinal form factor), we found

\[ \tilde{\Pi}_0^{(11)} = -\frac{N_c}{4\pi^2} \left( q_u q_d B^2 \right) \frac{\left[ q_{||}^2 + q_\perp^2 \right]}{q^2} \]

In the limit \( q_\perp^2 \to 0 \)

\[ \tilde{\Pi}_0^{(11)} q_\perp^2 \to 0 \Rightarrow -\frac{N_c}{4\pi^2} \left( q_u q_d B^2 \right) \]
For the other second order corrections (the notation is evident)

\[
\tilde{\Pi}_0^{(02)}(q^2) = -\frac{N_c}{24\pi^2} (q_u B)^2 \left[\frac{(q_\parallel^2 + q_\perp^2)}{q^2} + 2 \frac{q_\parallel^2 q_\perp^2}{q^4}\right]
\]

\[
\frac{q_\perp^2 \rightarrow 0}{\rightarrow} - \frac{N_c}{24\pi^2} (q_u B)^2
\]

\[
\tilde{\Pi}_0^{(02)}(q^2) = -\frac{N_c}{24\pi^2} (q_d B)^2 \left[\frac{(q_\parallel^2 + q_\perp^2)}{q^2} + 2 \frac{q_\parallel^2 q_\perp^2}{q^4}\right]
\]

\[
\frac{q_\perp^2 \rightarrow 0}{\rightarrow} - \frac{N_c}{24\pi^2} (q_d B)^2.
\]

Adding all these contributions

\[
\Pi_0^{B^2} = -\left(\frac{17}{18}\right) \frac{(eB)^2}{4\pi^2}
\]

And the Wilson coefficient in pQCD

\[
C_0^{(2)} = -\left(\frac{17}{18}\right) \frac{1}{4\pi^2}
\]
The last ingredient has to do with the magnetic field dependence of $f_\pi$.

The Gell-Mann-Oakes-Renner relation provides us with this information:

\[
\psi_5(q^2) = i \int d^4x e^{iqx} \langle 0 | T(\partial^\mu A_\mu(x) \partial^\nu A_\nu^\dagger(0)) | 0 \rangle
\]

This correlator implies:

\[
m^2_\pi f^2_\pi = -2(m_u + m_d) \langle \bar{q}q \rangle
\]

Which means:

\[
f^2_\pi = -2\beta \langle \bar{q}q \rangle.
\]

And:

\[
m^2_\pi = \frac{1}{\beta} (m_u + m_d).
\]
In fact, we have

\[
\frac{f_\pi^2(eB)}{f_\pi^2} = \frac{\langle \bar{q}q \rangle(eB)}{\langle \bar{q}q \rangle}
\]

Therefore, to a good approximation for small magnetic fields, the magnetic field dependence of the pion decay constant is determined by the quark condensate.
Solution to the Sum Rules

\[ s_0 = -8\pi B\langle \bar{q}q \rangle_{(eB)} \]
\[ C_4\langle O_4 \rangle = -2(eB)B\langle \bar{q}q \rangle_{(eB)} + 8\pi(B\langle \bar{q}q \rangle_{(eB)})^2 \]
\[ + \left( \frac{17}{18} \right) \frac{(eB)^2}{4\pi^2}. \]

The chiral perturbation constant is obtained using the physical pion and light-quark masses
The magnetic dependence of the quark condensate is provided by our friends of the lattice community:


\[
\frac{\langle \bar{q}q \rangle_{(eB)}}{\langle \bar{q}q \rangle} = 1 + a(eB) + b(eB)^2
\]

\[(a = 0.85 \text{ GeV}^{-2}, b = 0.34 \text{ GeV}^{-4})\]
Results
The result is interesting because, in a finite temperature scenario, there will be a competition between the thermal and the magnetic contributions to the evolution of the continuum threshold. This means that, in principle, the critical temperature will get a dependence on the magnetic field.

The magnetic field dependence on the continuum threshold is proportional to the quark condensate. The magnetic field both helps the formation of the condensate and act against deconfinement.
Thank You!!
Thermal Extension of the QCD Sum Rules

• There are important differences:

• 1) The vacuum is populated (a thermal vacuum)

• 2) A new analytic structure in the complex
  $s$-plane appears, due to scattering. This effect turns
  out to be very important
The current-current correlator in a thermal vacuum (a populated vacuum) corresponds to

\[ G^{R}_{\mu,\nu}(\omega, \vec{p}) = i \int d^4x \ e^{ipx} \theta(x^0) \langle \langle [J_\mu(x), J_\nu(0)] \rangle \rangle \]

\[ \langle \langle \cdots \rangle \rangle = \sum_n e^{-E_n/T} \langle n| \cdots |n \rangle \]
New cut associated to a scattering process with quarks (antiquarks) in the populated vacuum (Bochkarev-Schaposnikov)
The critical temperature seems to be the same for chiral restoration as well as for deconfinement!!