Agenda

1. EFT for Ultracold Atoms I: Effective Field Theories & Universality
2. EFT for Ultracold Atoms II: Cold Atoms & the Unitary Limit
3. EFT for Ultracold Atoms III: Weak Coupling at Finite Density
4. EFT for Ultracold Atoms IV: Few-Body Systems in the Unitary Limit
5. Beyond Ultracold Atoms: Halo Nuclei and Hadronic Molecules

Literature

Resolution of a Microscope

- Resolution of microscope is limited

- Diffraction effects:
  Light is EM wave ($\lambda \approx 550 \text{ nm}$)

- Resolution: $d = \frac{\lambda}{A_N}$

- Aperture: $A_N \approx 0.95...1.5$

  $\rightarrow d_{max} \approx 0.4 \mu m$

- Quantum mechanics:
  particles have wave character $\Rightarrow$ matter waves

  $\lambda = \frac{hc}{E}, \quad E \gg m$ (de Broglie, 1924) $\Rightarrow$ QM microscope
Hydrogen Atom

- Quantum bound state of $e^-$ and $p$

- Energy levels (lowest order): Schrödinger-Eq. for $V(r) = -e^2/r$
  
  $$E = E_0 = -\frac{m_e \alpha^2}{2n^2}, \quad \alpha = \frac{e^2}{4\pi}$$

- Corrections beyond leading order: $E = E_0 \left[ 1 + \mathcal{O}(\alpha, \frac{m_e}{m_p}, \ldots) \right]$
  
  - corrections from EM interaction: fine structure $\vec{L} \cdot \vec{S} \sim \alpha^4$, ...
  - corrections from proton structure: finite proton mass $m_p$
    - hyperfine structure: $\mu_p$
    - finite proton size: $r_p$
Dimensional Analysis

- Estimate effect of $r_p$ on $E_0$

- Natural scales: $a_0 = 1/(m_e \alpha) \sim 0.5 \, \text{Å}$

- Proton charge radius: $G_E(q^2) = 1 + q^2 \left( \frac{r_p^2}{6} \right) G_E'(0) + \ldots$

\[
G_E'(0) \sim \frac{1}{m_p^2}, \quad q \sim \frac{1}{a_0} \quad \Rightarrow \quad \left( \frac{m_e \alpha}{m_p} \right)^2 \sim 10^{-11}
\]

- Size of correction to $E_0$: $10^{-11} E_0 \rightarrow 10^{-11} \times 3 \cdot 10^{15} \, \text{Hz} \sim 30 \, \text{kHz}$

- Measurement of proton size in (muonic) hydrogen ($m_\mu/m_e \approx 200$)

- Proton radius puzzle:

\[
r_p|_{\mu H} = 0.84184 \pm 0.00067 \, \text{fm} \Leftrightarrow r_p|_{H} = 0.8779 \pm 0.0094 \, \text{fm}
\]


- How to construct an effective theory to calculate these corrections?
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Need to identify:
- Low and high scales in the problem: $M_{lo}, M_{hi}, ...$
- Active degrees of freedom: coordinates, particles, ...
- Symmetries to constrain interactions
- **Power counting:** organize the theory through expansion in $M_{lo}/M_{hi}$

**Guiding principle:**

Include long-range physics explicitly, parametrize short-range physics in low-energy constants

**Consider a classical example**
Example: Multipole Expansion

- Separation of scales: $\langle r' \rangle \ll r$
- Short-range physics: expand in $\frac{\langle r' \rangle}{r}$
  $\longrightarrow$ “power counting”
- Breakdown scale: $\langle r' \rangle \sim r$

$$\phi(\vec{r}) = \sum_{l,m} C_l Y_{lm}(\Omega) \int d^3 r' \rho_1(\vec{r}') Y^*_{lm}(\Omega') (r')^l$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l,m} C_l Y_{lm}(\Omega) Y^*_{lm}(\Omega') \frac{r^l_l}{r_{l+1}^{l+1}}$$
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$$\phi(\vec{r}) = \sum_{l,m} \frac{C_l Y_{lm}(\Omega)}{r^{l+1}} \left[ \int d^3 r' \rho_1(\vec{r}') Y_{lm}^*(\Omega')(r')^l + \int d^3 r'' \frac{\rho_2(\vec{r}'')}{|\vec{r} - \vec{r}'|} \right]$$

- Long-range physics: include explicitly
- Use these ideas in the framework of Quantum Field Theory
Quantum Field Theory is “only” way to satisfy quantum mechanics, Lorentz invariance, and cluster decomposition

⇒ to calculate the most general S-matrix consistent with given symmetries for any theory below some scale simply use the most general effective Lagrangian $\mathcal{L}_{\text{eff}}$ consistent with these principles in terms of the appropriate asymptotic states

(Weinberg, 1979)

Need “power counting scheme” to organize the infinitely many terms in $\mathcal{L}_{\text{eff}}$

⇒ predictive power
Construction of an QFT

1. Construct $\mathcal{L}$ respecting symmetries (e.g. gauge invariance of QED)

\[ \psi \rightarrow \psi' = e^{i\alpha(x)}\psi, \quad A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \alpha_\mu(x) \]

2. Retain renormalizable interactions ($D \leq 4$):

\[ [\psi] = \frac{3}{2}, \quad [A] = 1 \]

- keep $\bar{\psi}\gamma_\mu \psi A^\mu$, 
- but drop $\bar{\psi}\sigma_{\mu\nu} \psi F^{\mu\nu}$, 
- $(F^{\mu\nu}F^{\mu\nu})^2$

\[ \Rightarrow \quad \mathcal{L} = \bar{\psi}(i\partial - eA - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \]

3. Calculate Feynman diagrams:

\[ \begin{array}{c}
\Rightarrow \\
\mu_\epsilon = \frac{-e g_e S_e}{2m_e}, \quad g_e = 2 \left[ 1 + \frac{e^2}{8\pi^2} + \mathcal{O}(e^4) \right]
\end{array} \]
Renormalizability

- **Renormalization**: method to make sense of infinities in Quantum Field Theories
  - Some observables are sensitive to short-distances ⇒ match to experiment and predict other observables
  - **Classical renormalizability**: redefinition of a finite number of terms sufficient to all orders (QED: $e, m_e, Z_3, Z_2$)
  - **EFT**: new structures appear at every order

- **Renormalizable theories have been very successful** ⇒ **Standard Model (SM)**

- **Non-renormalizable interactions** ($D > 4$) are suppressed at low energies ← factors $1/M_{hi}^{D-4}$

- **Modern understanding**: SM is also low-energy EFT, physics beyond SM can be included through non-renormalizable interactions
Construction of an EFT

- Construct most general $\mathcal{L}_{\text{eff}}$ respecting underlying symmetries

- Exploit separation of scales: $E \ll E_c$

- Non-renormalizable theory, but only finite number of operators in $\mathcal{L}$ contribute at each order
  $\Rightarrow$ Low-energy constants (LEC)

- Symmetries and light dof’s must be known

- Work at low energies: $E \ll E_c$

- Fix LEC’s from matching (to experiment, other theory, ...)

- Calculate observables in expansion in $E/E_c$
  $\Rightarrow$ limited range of applicability
Light-By-Light Scattering

- **Classic Example** (Euler, Heisenberg, 1936)
- Photon energy $\omega$, electron mass $m_e$
- Separation of scales: $\omega \ll m_e$
  $\Rightarrow$ theory simplifies
  $\Rightarrow \mathcal{L}_{QED}[\psi, \bar{\psi}, A_\mu] \rightarrow \mathcal{L}_{\text{eff}}[A_\mu]$

- Invariants: $F_{\mu\nu}F^{\mu\nu}$, $(F_{\mu\nu}F^{\mu\nu})^2$, $(F_{\mu\nu}\tilde{F}^{\mu\nu})^2$, ...

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \frac{e^4}{360\pi^2m_e^4} \left[ (\vec{E}^2 - \vec{B}^2)^2 + 7(\vec{E} \cdot \vec{B})^2 \right]$$

- Energy expansion: $(\omega/m_e)^{2n}$
- Cross section: $\sigma(\omega) = \frac{1}{16\pi^2} \frac{973}{10125\pi} \frac{e^8}{m_e^2} \frac{\omega^6}{m_e^6} + \mathcal{O}(\omega^8)$
- Bounds from high-power lasers (Della Valle et al., PRD 90 (2014) 092003)
Fermi Theory

- Weak decays
  - mediated by $W^\pm$ bosons ($M_W \approx 80$ GeV)
  - energy release in neutron $\beta$-decay $\sim 1$ MeV [$n \to p e^- \bar{\nu}_e$]
  - energy release in kaon decays $\sim 300$ MeV [$K \to \pi e\nu$]

\[
\frac{e^2}{8 \sin^2 \theta_W} \times \frac{1}{M_W^2 - q^2} \quad q^2 \ll M_W^2 \quad \rightarrow \quad \frac{e^2}{8 M_W^2 \sin^2 \theta_W} \left\{ 1 + \frac{q^2}{M_W^2} + \ldots \right\} = \frac{G_F}{\sqrt{2}} + \ldots
\]

$\Rightarrow$ Fermi Theory of Weak Interaction
Structure of EFTs

- **Energy → momentum expansion** (derivatives acting on fields)
- **Dimensional analysis** (breakdown scale $\Lambda$)
  - derivatives $\rightarrow$ powers of typical momentum $q$
  - one derivative: $\partial \sim q/\Lambda$
  - vertex with $N$ derivatives: $\partial^N \sim (q/\Lambda)^N$
  - terms with more derivatives are suppressed if $q \ll \Lambda$
- **Loops**: additional powers of momenta from vertices and loop momenta

$\implies$ loops are generally suppressed compared to trees (exceptions!)

$\sim (q/\Lambda)^N$  $\sim (q/\Lambda)^{2N}$

$\int d^4pf(p, q)$
**Universality**: Physical systems with different short-distance behavior exhibit identical behavior at large distances
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Condensed matter systems near critical point

\[ \rho_{\text{liq/gas}}(T) - \rho_c \rightarrow \pm A(T_c - T)^\beta \]

liquid-gas system

\[ M_0(T) \rightarrow A'(T_c - T)^\beta \]

Ferromagnet (one easy axis)

Universality class determines critical exponents: \( \beta = 0.325 \)
Universality and Pointilism

- Painting at the limit of resolution of the human eye

G. Seurat, A Sunday on La Grande Jatte
Universality and Pointilism

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Renormalization Group

- **RG:** Important concept in many areas of physics (Gell-Mann, Low, Kadanoff, Wilson,..)
  - Critical phenomena, evolution of coupling constants, Fermi liquid theory, ...
- $\mathcal{L} \Rightarrow \text{point } g = (g_1, g_2, \ldots )$ in space of coupling constants
  \[ \mathcal{L} = \sum_n g_n O_n \]
- **RG transformation** $\Rightarrow$ integrate out high-energy modes
  $\Rightarrow$ change of resolution scale $\Rightarrow$ couplings flow $\Rightarrow$ generate EFT
- **RG flow expressed by differential equation for** $g$:
  \[ \Lambda \frac{d}{d\Lambda} g = \beta(g) \]
Fixed Points

- RG fixed point: $\beta(g_*) = 0 \implies \Lambda \frac{d}{d\Lambda} g_* = 0$

- Universal properties
  - Scale invariance $\iff$ power law behavior

- Linearise: $\Lambda \frac{d}{d\Lambda} g = B \cdot (g - g_*) \implies g - g_* = \Lambda^B \cdot c$

$\implies$ Critical exponent: $B$
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M. Frame, Yale
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M. Frame, Yale
Limit Cycles

- **RG limit cycle:** (Wilson, 1971)
  - Family \( \mathcal{L}_* \) of Lagrangians: \( g(\Lambda) = g(\mu^n \Lambda) \)
  - \( \mathcal{L}_*(\theta) \) parametrized by angle \( \theta \propto \ln \Lambda \)

- **Complex critical exponent:** \( g = \text{Re} \left[ \Lambda^{2\pi i/\ln \mu} \right] = \cos(2\pi \ln \Lambda / \ln \mu) \)
  - Discrete scale invariance \( \leftrightarrow \) Log-periodic behavior
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Russian Dolls
Low-Energy Universality

- **Renormalization Group:** Systems with very different fundamental interactions can behave similarly at low energies

  \[ \implies \text{Universal properties} \]

- **Nuclear Physics:**
  - Nucleons and pions

- **Atomic Physics:**
  - Born-Oppenheimer plus van der Waals
  - At sufficiently low energy: short-range interactions

- **Contact interactions**
Summary

- Short-distance physics can not be resolved at low energies
- Effective (Field) Theory: general method to exploit separation of scales
  \[ \implies \text{power counting in } M_{lo}/M_{hi} \]
- Renormalizable and non-renormalizable field theories
- Universality: systems with different short-distance physics can behave identically at long distances
  \[ \implies \text{Renormalization group} \]
- Fixed points \iff Limit cycles
- Scale invariance \iff discrete scale invariance