Complexity driven collapses in large random economies

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A tale of two phase transitions

Financial market

Industrialized economy

Complexity

\( \langle s^* \rangle \)

\( \phi \)

\( X_W \)

\( \pi/n = 0.05, f/n = 0.05 \)

\( \pi/n = 0.05, f/n = 0.15 \)

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Complexity driven collapses in large random economies
Before we start

★ **Goal**: show that complexity in financial / economic environments can lead to unexpected (catastrophic) consequences

★ **Where**: stylized models that most mainstream economists use and believe in

★ **How**: “promoting” key independent variables to random variables

★ **Benefits**: analytical access to typical properties

★ **Disclaimer**: technically not “our” results!
Collaborators

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Financial instability from local market measures

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Part I: Modeling the emergence of arbitrage opportunities

★ Financial institutions need to price a very large number of instruments every day

★ Different institutions use different models and calibrate them on different data
  ★ Models have to be simple \(\Rightarrow\) \textit{unavoidable approximations}

★ It has been argued that exceedingly different models and/or calibration datasets might lead to a system of inconsistent prices, which in turn might lead to:
  ★ \textit{Arbitrage opportunities}
  ★ Trading activity that induces \textit{further misalignment} (self-sustaining bubbles)


★ This talk: stylized statistical model to link heterogeneity in pricing models and emergence of arbitrage opportunities
One period economies

Ingredients:

- Set of $N$ financial instruments
- Initial time $t = 0$ and final time $t = 1$
- Set $\Omega$ of possible time 1 states of the world (unknown at time 0)

- Risk neutral probability measure (RNPM) $q$ on $\Omega$: $\sum_{\omega=1}^{\Omega} q^\omega = 1$

\[
q^\omega > 0 \quad \text{for all } \omega \in \Omega
\]

\[
s_i(0) = \sum_{\omega=1}^{\Omega} q^\omega s_i^\omega(1) = \langle s_i(1) \rangle_q \quad \text{for } i = 1, \ldots, N
\]

- RNPMs allow to compute the fair price of financial instruments
One period economies

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- RNPMs allow to compute the fair price of financial instruments
Formal setup

- **Portfolio**: \( V(0) = \sum_{i=1}^{N} w_i s_i(0) \rightarrow V^\omega(1) = \sum_{i=1}^{N} w_i s_i^\omega(1) \)
- **Arbitrage opportunity**: a portfolio investment strategy such that

\[
V^\omega(1) \geq V(0) \quad \text{for all } \omega \in \Omega
\]
\[
V^{\omega'}(1) > V(0) \quad \text{for at least one } \omega'
\]

**THEOREM**: no arbitrage opportunities if and only if there exists a RNPM on \( \Omega \).

**How many RNPMs are there?**

For a given system of prices we need to solve a system of \( N \) equations in \( \Omega \) unknowns

\[
\sum_{\omega=1}^{\Omega} q^\omega s_i^\omega(1) = s_i(0)
\]

- If \( \Omega = N \): one solution (not necessarily \( q^\omega > 0 \ \forall \omega \))
- If \( \Omega > N \): infinitely many
- If \( N > \Omega \): none
The arbitrage region

- **Assumption:** each financial instrument is priced according to a specific probability measure (not necessarily a RNPM)

\[
\text{instrument } i \longleftrightarrow \text{pricing measure } q_i^\omega
\]

\[
s_i(0) = \langle s_i(1) \rangle q_i = \sum_{\omega=1}^{\Omega} q_i^\omega s_i^\omega(1)
\]

- **Question:** when do RNPMs exist for such a market?

- **Solution:** computing the volume of the arbitrage region, i.e. region in the \(N\)-dimensional space of portfolio weights \(\{w_1, \ldots, w_N\}\) such that for any state \(\omega\)

\[
V^\omega(1) - V(0) \geq 0 \implies \sum_{i=1}^{N} w_i(s_i^\omega(1) - s_i(0)) \geq 0
\]

- **Arbitrage region volume:**

\[
V = \int_{-\infty}^{+\infty} dw_1 \ldots \int_{-\infty}^{+\infty} dw_N \prod_{\omega=1}^{\Omega} \Theta \left( \sum_{i=1}^{N} w_i(s_i^\omega(1) - s_i(0)) \right)
\]
A random geometry approach

- Computing the arbitrage region volume for an individual market is hard

- **Solution**: consider an ensemble of markets by promoting returns and pricing measures to random variables

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**Statistical Physics of disordered systems**

- We look for **self-averaging quantities** in the limit $N, \Omega \to \infty$ with $n = N/\Omega$ fixed
- Self-averaging quantities are typically intensive quantities, we expect $V \sim \ell^N$ so we compute

$$\frac{1}{N} \langle \log V \rangle_{s,q}$$

- Using replicas we find a general solution that depends on the statistical properties of the returns and pricing measures
Case study: subset of states

* Each instrument is priced only on a subset $\Omega_K \subset \Omega$ states made of $K$ states

$$q_\omega^i = \begin{cases} 
  1/K & \text{if } \omega \in \Omega_K \\
  0 & \text{if } \omega \notin \Omega_K
\end{cases}$$

* Gaussian returns: $s_\omega^i(1) - s_\omega^i(0) \sim N(0, 1)$

* $k = K/\Omega$ measures heterogeneity between pricing measures
Proof of concept: a system of exceedingly different pricing measures may give rise to arbitrage opportunities (financial instabilities)

Complexity $\Rightarrow$ Arbitrage $\Rightarrow$ Speculative investments

Results support the call by Albanese et al. for global pricing measures!

Part II: Micro(economics) $\longrightarrow$ Macro(economics)

- **General problem**: aggregating microeconomic behaviour and interactions between economic agents into macroeconomic laws

- **Specific problem**: understanding the macroeconomic behaviour of modern industrialized economies

- **Input-output analysis**: understanding the linkages and mutual impact between different productive entities

  output from sector / firm $A$ $\longrightarrow$ input to sector / firm $B$
Input-output analysis (I)

Pre-industrial era: Tableaux économiques (1758)

Figure: François Quesnay (1694-1774)
Economies have evolved to remarkable levels of complexity: the production of technologically sophisticated objects involves multiple production processes feeding each other, often delocalized / outsourced across multiple firms.

Modern input-output models are written in the language of General Equilibrium (GE) Theory:

- **Profit maximizing firms** ↔ **utility maximizing consumers** ↔ **market prices**
A GE input–output model (I)

Ingredients:

- $C$ goods
  - Raw goods: $x_0^c = 1$
  - Consumer goods: $k^c = 1$

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<th>$x_0^c = 1$</th>
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- $N$ technologies / firms
  - $q_i^c < 0$: good $c$ is an input to technology $i$
  - $q_i^c > 0$: good $c$ is an output of technology $i$
A GE input–output model (II)

**Goal:** determining equilibrium prices $p^c$ such that

- Firms maximize profits $r_i$ by setting scales of production $s_i \geq 0$

$$r_i = s_i \sum_{c=1}^{C} p^c q_i^c$$

- Consumers maximizes utility compatibly with a budget constraint

$$U(x) = \sum_{c=1}^{C} k^c u(x^c)$$

- Consumption

$$x^c = x_0^c + \sum_{i=1}^{N} s_i q_i^c \geq 0$$

Initial endowment

Net production
GE meets Complexity

★ Problems of standard GE approach:
   1) No heterogeneity (representative consumer)
   2) Reliance on precise knowledge of the economy’s input–output matrix \( q_i^c \)

★ Solution: “promoting” \( q_i^c \) and goods’ labels \( x_0^c, k^c \) to random variables

Systems with random interactions

F. Dyson: “What is here required is a new kind of statistical mechanics, in which we renounce exact knowledge not of the state of the system but of the system itself.”

★ Methods: averaging over ensemble of economies in the limit \( N, C \to \infty \) with \( n = \frac{N}{C} \) fixed → Averaging over all possible input–output matrices and all possible representative agents

★ Analytical access to the economy’s typical properties
The economy’s phase diagram

\[ n = \frac{N}{C} = \frac{\text{# of technologies}}{\text{# of goods}} \]

\[ \pi = \frac{\text{# raw goods}}{\text{# of goods}} \]

\[ \langle s^* \rangle = \text{average scale of production} \]

**Result:** Transition from no-industrialization trap to industrial production happens without need to invoke a “big push”

**Result:** Introduction of new technologies has different impacts in different regimes

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Economic expansion through intermediate goods

- Economies can expand via outsourcing / externalization, i.e. the creation of new markets for intermediate goods
- Increase in the number of firms $\rightarrow$ increase in the number of interactions through market prices $\rightarrow$ increase in complexity!
Complexity driven collapse of equilibria

\[
\langle s^* \rangle_{0,0.05, 0.05}, \langle s^* \rangle_{0,0.15, 0.05}, \langle s^* \rangle_{0,0.05, 0.15}, \langle s^* \rangle_{0,0.15, 0.15}
\]

\[
\phi = \frac{\text{fraction of active firms}}{\text{fraction of intermediate goods}}
\]

\[
X_W = \text{waste}
\]

\* Result: expansion of the economy via externalization / outsourcing leads to a sudden shutdown
Nature of the phase transition

* Optimization problem for consumers

\[
\max_{\{s \geq 0\}} U(x) \quad \text{where} \quad x^c = x_0^c + \sum_{i=1}^N s_i q_i^c \geq 0
\]

* Volume of possible solution space

\[
V = \int_0^\infty ds_1 \ldots \int_0^\infty s_N \prod_{c=1}^C \Theta \left( x_0^c + \sum_{i=1}^N s_i q_i^c \right)
\]

* Same problem as the arbitrage volume!
Remarks

- Full solution of GE in a simple input-output setting
- NOT our predictions → GE’s predictions!
- Analytical treatment of combinatorial phase transition in high-dimensional geometry (see Donoho and Tanner (2009))

Figure 4 Greece: Real GDP (2005=100)

Sources: IMF; EC; authors’ calculations