Representation theory of Lie algebra of vector fields on a torus

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Let \( d_k = t_k \frac{\partial}{\partial t_k} \). Cartan subalgebra: \( \{d_1, \ldots, d_N\} \)
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\[ W_N = \text{Der} A = \bigoplus_{\rho=1}^{N} \mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] \frac{\partial}{\partial t_\rho}. \]

Let \( d_k = t_k \frac{\partial}{\partial t_k} \). Cartan subalgebra: \( \{d_1, \ldots, d_N\} \) induces \( \mathbb{Z}^n \)-grading on \( W_N \).
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• Harish-Chandra modules
Theorem (E. Rao, 1996)

Let $U$ be an irreducible finite-dimensional $\mathfrak{gl}(N)$-module. Then

$$V = \mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] \otimes U$$

is an irreducible $W_N$-module,
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Irreducible tensor modules

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$$V = \mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] \otimes U$$

is an irreducible $W_N$-module, unless it appears in the de Rham complex:

$$\Omega^0(\mathbb{T}^N) \to \Omega^1(\mathbb{T}^N) \to \ldots \to \Omega^N(\mathbb{T}^N).$$
**Definition**

An $AW_N$-module $V$ is a module for $W_N$, and simultaneously a module for the commutative unital algebra $A$, with the two actions being compatible:

$$\eta(fv) = \eta(f)v + f(\eta v),$$

for $\eta \in W_N$, $f \in A$, $v \in V$. 
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**Theorem (Y.B., 2004)**

The category of $AW_N$-modules with finite-dimensional weight spaces and weight lattice $\mathbb{Z}^N$ is equivalent to the category of finite-dimensional $\mathcal{L}_+$-modules.
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Let $U$ be a finite-dimensional $\mathcal{L}_+$-module. Then $\mathbb{C}[x_1, \ldots, x_N] \otimes U$ admits the action of $W_N$:

$$t^s d_a \left( t^m \otimes u \right) = m_a t^{m+s} \otimes u + \sum_{k \in \mathbb{Z}^N \setminus \{0\}} \frac{s^k}{k!} t^{m+s} \otimes \rho \left( x^k \frac{\partial}{\partial x_a} \right) u.$$
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**Theorem (B. - Futorny, 2013)**

For a cuspidal $W_N$-module $V$ satisfying $W_N V = V$, there exists a cuspidal $AW_N$-module $M$ with a surjective homomorphism of $W_N$-modules:

\[ \pi : M \rightarrow V. \]
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Every irreducible Harish-Chandra module for $W_N$ is either
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or
(2) a module of a highest weight type, induced from a tensor module of rank $N - 1$, 

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Theorem (B. - Futorny, 2013)

Every irreducible Harish-Chandra module for $W_N$ is either
(1) a quotient of a tensor module,
or
(2) a module of a highest weight type, induced from a tensor module of rank $N - 1$, and twisted with an automorphism from $GL_N(\mathbb{Z})$. 
Consider $W_{N+1} = \text{Der} \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_N^{\pm 1}]$
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W_{N+1}^0 = W_N \oplus \mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] d_0.
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Let \( V \) be a tensor module for \( W_N \) with the action of \( t^s d_0 \) by \( \alpha t^s \),
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\alpha \in \mathbb{C}.
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Consider $W_{N+1} = \text{Der} \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_N^{\pm 1}]$ with $\mathbb{Z}$-grading by $d_0$.

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Let $V$ be a tensor module for $W_N$ with the action of $t^s d_0$ by $\alpha t^s$, \(\alpha \in \mathbb{C}\).

Let $W_{N+1}^+ V = (0)$ and consider the induced module

\[
M(V) = U(W_{N+1}^-) \otimes V.
\]
Theorem (Berman - B., 1999)

(1) $M(V)$ has a unique maximal submodule $M^{rad}$.
Irreducible modules of the highest weight type

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(1) $M(V)$ has a unique maximal submodule $M^{\text{rad}}$.

(2) $L(V) = M(V)/M^{\text{rad}}$ is an irreducible Harish-Chandra module.
Let $V_{Hyp} = \mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] \otimes \mathbb{C}[u_{pj}, v_{pj} | p = 1, \ldots, N, j \in \mathbb{N}]$. 
Let $V_{Hyp} = \mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] \otimes \mathbb{C}[u_{pj}, v_{pj} \mid p = 1, \ldots, N, j \in \mathbb{N}]$.

Let $V_{\hat{gl}_N}$ be the $\hat{gl}_N$ vertex algebra at level 1.
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Let $V_{\hat{gl}_N}$ be the $\hat{gl}_N$ vertex algebra at level 1.

Let $V_{Vir}$ be the Virasoro vertex algebra with central charge 0.
Theorem (B. - Futorny, 2011)

Let $L_{\text{Hyp}}$ be a module for $V_{\text{Hyp}}$, $L_{\hat{gl}_N}$ be a module for $V_{\hat{gl}_N}$ and $L_{\text{Vir}}$ be a module for $V_{\text{Vir}}$.

Then $L = L_{\text{Hyp}} \otimes L_{\hat{gl}_N} \otimes L_{\text{Vir}}$ admits the action of $W_{N+1}$.

If $L_{\text{Hyp}}$, $L_{\hat{gl}_N}$ and $L_{\text{Vir}}$ are irreducible then $L$ is an irreducible $W_{N+1}$-module of the highest weight type, unless $L$ appears in the chiral de Rham complex.
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Then

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L = L_{\text{Hyp}} \otimes L_{\hat{gl}_N} \otimes L_{\text{Vir}}
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admits the action of \( W_{N + 1} \).

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Chiral de Rham complex

[Malikov-Shekhtman-Vaintrob, 1998] Let $V_{\mathbb{Z}^N}$ be the lattice vertex superalgebra. Then

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$$\ldots \rightarrow \Omega_{ch}^{-2} \rightarrow \Omega_{ch}^{-1} \rightarrow \Omega_{ch}^{0} \rightarrow \Omega_{ch}^{1} \rightarrow \ldots$$
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**Theorem (B. - Futorny, 2011)**

*Each component of the chiral de Rham complex is a $W_{N+1}$-module, and each differential is a homomorphism of $W_{N+1}$-modules.*