Few-body models for nuclear astrophysics

Y. Suzuki (Niigata Univ. & RIKEN)

Ab initio calculation of four-nucleon reactions
Reactions of importance for $^4$He generation

d+d reactions $^2$H(d,p)$^3$H, $^2$H(d,n)$^3$He, $^2$H(d,γ)$^4$He

Challenges to a fair calculation

$^{12}$C and $^{16}$O generations

Triple-α reaction $\alpha+\alpha+\alpha \rightarrow ^{12}$C
Radiative capture reaction $^{12}$C(α,γ)$^{16}$O
Dr. Aristides de Azevedo Pacheco Leão

6-9.1978 UFPE (Recife) First exchange researcher by JSPS&BAS agreement

Dr. Aristides de Azevedo Pacheco Leão
Nuclear Landscape

- stable nuclei
- known nuclei
- drip line
- r-process
- terra incognita
- neutron stars

Protons vs. Neutrons graph showing:
- 288
- 3000
- ~4000
- \(O(10^{57})\) nucleons
More unexplored nuclei than known nuclei

Nuclear structure models established so far are mostly based on studies of stable nuclei

**Now intensive studies of unstable nuclei**

how to produce nuclei far from stability
what properties (size, mass, etc.)
textbook knowledge may not be valid here
very weakly bound nucleons
neutron-halo structure
thick neutron skin
breakdown of magic number

Developing a many-body theory for describing both stable and unstable nuclei is interesting and very important

**Nuclear force is complicated and still not fully understood**

importance of pion exchange at medium and large NN distances
strong repulsion at short NN distance
non-central component, spin-dependent
(nucleons are composites of quarks and gluons)
Where and how are nuclei (elements) formed?

‘Nuclear astrophysics’

Interdisciplinary branch of nuclear physics, astrophysics, …

H. Bethe (1906-2005)
Contribution to the theory of nuclear reactions, esp.
discoveries concerning the energy production in stars, Nobel prize 1967

W. A. Fowler (1911-1995)
Studies of the nuclear reactions of importance in the
formation of the chemical elements in the universe, Nobel prize 1983

F. Hoyle (1915-2001)
Coining the phrase of Big Bang
Triple-alpha process (Hoyle resonance in $^{12}$C, 1953)
Supernova nucleosynthesis, 1954
Three main scenarios

**Big Bang nucleosynthesis:**
Dominant reactions in primordial nucleosynthesis
Few reactions produce elements up to $A=7$, $A=5$, 8 gap

**Stellar nucleosynthesis**
Gravitational contraction of star (depend on mass and temperature, etc.)
pp chain and CNO cycle that convert H to He, He burning, …, up to Fe
$4p \rightarrow ^4\text{He} + 2e^- + 2\nu$

**Supernova nucleosynthesis**
Synthesis of elements heavier than Fe in supernova explosion (F. Hoyle 1954)
Neutron star merger
s process, r process, …
Big Bang nucleosynthesis

T \sim 10^9 \text{ K} \sim 90 \text{ keV}
Generation of \( d \)
\( p+n \longrightarrow d+\gamma \)

1: \( n \leftrightarrow p \)
2: \( p(n, \gamma)d \)
3: \( d(p, \gamma)^3\text{He} \)
4: \( d(d, n)^3\text{He} \)
5: \( d(d, p)^3\text{H} \)
6: \( ^3\text{H}(d, n)^4\text{He} \)
7: \( ^3\text{H} (^4\text{He}, \gamma)^7\text{Li} \)
8: \( ^3\text{He}(n, p)^3\text{H} \)
9: \( ^3\text{He}(d, p)^4\text{He} \)
10: \( ^3\text{He}(^4\text{He}, \gamma)^7\text{Be} \)
11: \( ^7\text{Li}(p, ^4\text{He})^4\text{He} \)
12: \( ^7\text{Be}(n, p)^7\text{Li} \)

\( d(d, \gamma)^4\text{He} \)
**Deuteron:** small binding energy, easily broken non-vanishing Q moment (deformed) tensor force due to pion

Tensor operator \[ S_{12} = 2 \left[ 3 \frac{(\vec{S} \cdot \vec{r})^2}{r^2} - \vec{S}^2 \right] \]

**AV8’ potential**  

\[ <L=0|\text{Tensor}|L=0> = 0, \quad <L=0|\text{Tensor}|L=2> \neq 0, \quad <S=0|\text{Tensor}|S=0> = 0 \]

\[ |d: 1^+ > = |L=0, S=1> + |L=2, S=1> \quad (5-6\% \text{ D-wave mix}) \]
Why are d+d reactions interesting?

Four-nucleon system may be small enough for ab initio calculations
Because of complexities of nuclear force, no fully microscopic calculations have been performed before … challenging
A special sensitivity of d+d reactions to tensor force

Niigata-ULB collaboration (K.Arai,S.Aoyama,Y.S.,P.Descouvemont,D. Baye)

Boson symmetry of d+d channel

Channel spin \((I=1+1)\) \(I = 0, 2\) (S) and 1 (A)
Two-d exchange \(I + l = \text{even}\)
Total angular momentum \(J = I + l\)

S-wave \((l=0)\) \(J=0, 2\); P-wave\((l=1)\) \(J=0,1,2\)
(positive parity); (negative parity)

Initial S-wave dominates astrophysical reactions
S-wave d+d entrance is possible only for \(J^\pi = 0^+, 2^+\)
Variational trial wave function = Superposition of correlated Gaussians
Many variational parameters have to be chosen judiciously (SVM)

Y.S., K.Varga, Lecture Notes in Physics, m54 (Springer, 1998)

AV8’ potential

<table>
<thead>
<tr>
<th></th>
<th>$^3\text{He}(\frac{1}{2}^+)$</th>
<th>$^4\text{He}(0^+)$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>GVR</td>
<td>PWE</td>
</tr>
<tr>
<td>$E$ (MeV)</td>
<td>-7.10</td>
<td>-7.10</td>
</tr>
<tr>
<td>$\langle T \rangle$</td>
<td>46.68</td>
<td>46.67</td>
</tr>
<tr>
<td>$\langle V_c \rangle$</td>
<td>-22.00</td>
<td>-21.98</td>
</tr>
<tr>
<td>$\langle V_{\text{Coul}} \rangle$</td>
<td>0.65</td>
<td>0.65</td>
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<tr>
<td>tensor</td>
<td>$\langle V_t \rangle$</td>
<td>-30.47</td>
</tr>
<tr>
<td>spin-orbit</td>
<td>$\langle V_b \rangle$</td>
<td>-1.97</td>
</tr>
<tr>
<td>$\sqrt{\langle r^2 \rangle}$ (fm)</td>
<td>1.79</td>
<td>1.79</td>
</tr>
</tbody>
</table>

Expt. -7.72 -28.30

To account for nuclear binding, three-nucleon force (TNF) is needed
Fitting data with TNF

<table>
<thead>
<tr>
<th></th>
<th>(d)</th>
<th>(^3\text{H})</th>
<th>(^4\text{He})</th>
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</thead>
<tbody>
<tr>
<td>(E) (MeV)</td>
<td>-2.24</td>
<td>-8.41</td>
<td>-28.43</td>
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<tr>
<td>(\sqrt{\langle r^2 \rangle}) (fm)</td>
<td>1.96</td>
<td>1.70</td>
<td>1.45</td>
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<tr>
<td>(P(L = 0))</td>
<td>94.23</td>
<td>91.25</td>
<td>85.56</td>
</tr>
<tr>
<td>(P(L = 2))</td>
<td>5.77</td>
<td>8.68</td>
<td>14.07</td>
</tr>
<tr>
<td>(P(L = 1))</td>
<td>-</td>
<td>0.07</td>
<td>0.37</td>
</tr>
</tbody>
</table>

Consequence of D-state probabilities in \(d\) and \(^4\text{He}\) for low-energy reactions

- If \(l = 0\) (S-wave entrance), main to main transition occurs in \(\Delta L = 0\) (no role of tensor force)
- Main to admixture or admixture to main transition occurs in \(\Delta L = 2\)
What we want to calculate

Reaction rates $<\sigma v>$ at temperature $T$:  
\[
N_A \langle \sigma v \rangle = N_A \left( \frac{8}{\pi \mu m_N (k_B T)^3} \right)^{\frac{1}{2}} \int \sigma(E) E \exp(-E / k_B T) dE
\]

Star center assumed as a perfect gas following the Maxwell-Boltzmann distribution

Stellar energy is much lower than Coulomb barrier  
Experimental measurement is very difficult  
$\sigma(E)$ strongly depends on $E$ (barrier penetration, S-wave)  
\[
\sigma(E) \sim \exp(-2\pi \eta)/E. \quad \eta: \text{Sommerfeld parameter}
\]

Astrophysical S factor  
\[
S(E) = \sigma(E) E \exp(2\pi \eta)
\]

Slow energy dependence
Transfer reactions

Realistic potentials reproduce astrophysical S-factors very well
MN potential gives too small values
Which $J^\pi$ does mostly contribute?

$J^\pi = 0^\pm, 1^\pm, 2^\pm$ states are included

K. Arai et al., PRL 107 (2011)
Decomposition of cross sections

$^2\text{H}(d, p)^3\text{H}$

2$^+$ contribution is largest below 60 keV
Among 6 paths for $J^\pi = 2^+$, $^5S_2 \rightarrow ^3D_2$ dominates at low energies $|L=0, S=2\rangle$ to $|L=2, S=1\rangle$ transition  Tensor force is responsible
With increasing energy, D-wave makes a larger contribution
Radiative capture \(^2\text{H}(d,\gamma)^4\text{He}\) at astrophysical energy

At extremely low energy S-wave is predominant \(\rightarrow J^\pi=0^+, 2^+

0^+ \rightarrow 0^+\) EM transition is forbidden

2\(^+\) \rightarrow 0^+\) EM transition: E2 (electric quadrupole transition)

If no tensor force \(\rightarrow\) No D states in d and \(^4\text{He}\)

\(\rightarrow\) No S-wave E2 transition

AV8’ reproduces S-factor, but MN fails at \(E < 0.3\text{MeV}\)

Usually E1 is dominant, but here it is small
**Triple-α reactions at low energy**

**Synthesis of $^{12}\text{C}$ element (the fourth abundant element)**

Hydrogen burning creates $^4\text{He}$
Because of mass gap at $A=5$ and 8,
$^{12}\text{C}$ synthesis is blocked

A way out is He burning by triple-α process
(Salpeter 1952)
First step: resonance formation $\alpha + \alpha \rightarrow ^8\text{Be}(0^+)$
$\tau \sim 10^{-16}$ s $>>$ transit time $10^{-19}$ s at 92 keV
Second step: radiative capture $^8\text{Be}(\alpha,\gamma)^{12}\text{C}$

Hoyle’s prediction (1953)
To explain $^{12}\text{C}$ abundance, the second step should proceed via S-wave resonance near $\alpha + ^8\text{Be}$ threshold
The resonance confirmed experimentally (Cook et al. 1957)

$^0+ \rightarrow ^{12}\text{C}(0^+) \text{ EM transition is forbidden}$
$3\alpha (0^+) \text{ continuum} \rightarrow ^{12}\text{C}(2^+) \text{ E2 transition}$
resonant and non-resonant contributions

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**Diagram:**

- $^8\text{Be}$ levels and transitions:
  - $^8\text{Be}(0^+) \rightarrow ^{12}\text{C}(0^+)$
  - $^8\text{Be}(2^+) \rightarrow ^{12}\text{C}(2^+)$
  - $3\alpha (0^+) \rightarrow ^{12}\text{C}(2^+)$

- $\Gamma=6 \text{ eV}$
- $\Gamma=8 \text{ eV}$
- $E=7367 \text{ keV}$
- $E=7654 \text{ keV}$
- $E=379 \text{ keV}$
- $E=92 \text{ keV}$

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Efimov, PLB33(1970)
At large distances:
- Coulomb potential barrier
- Centrifugal barrier

At shorter distances:
- Nuclear potential

Need to describe large-scale dynamics at low energies near $3\alpha$ threshold

Three charged-particles in continuum --- Notoriously difficult problem because they interact even at large distances

Asymptotics of three charged-particles is unknown
Hyperspherical coordinate method


No. of coordinates for 3-body system
3 × 3 − 3 = 6
One radial coordinate (hyperradius, $R$)
Five angle coordinates (hyperangles, $\Omega$)

10 lowest adiabatic hyperspherical potential curves for $J^\pi=0^+$
At large $R$, black curve dominated by $^8\text{Be}+\alpha$ channel successively has avoided crossings with 3$\alpha$ continuum curves

Complex absorbing potential at large $R$

Narrow width of Hoyle state
$\Gamma/E_{\text{res}} \approx 10^{-5}$
Calculation of the reaction rate in progress
Radiative capture reaction $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$

Determines the abundance ratio C/O=0.6
Plays an important role for stellar evolution

E1 and E2 transitions to the ground state from $^{12}\text{C}+\alpha$ continuum with $E \approx 300$ keV
Experimental measurement is extremely hard

**Points to be noted:**

Non-resonant transitions but continuum states with 1$^-$ and 2$^+$ are influenced by the subthreshold 1$^-$ and 2$^+$ states as well as the 1$^-$ resonant state with large $\alpha$ width
For E1 transition, isospin impurity has to be considered

The relevant states have structure of both shell-model and $^{12}\text{C}+\alpha$ cluster components
A unified description is desirable
NACRE compilation for $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$

C.Angulo et al., NPA656 (1999)
\[\text{\^{12}C+\textit{p}+\textit{p}+\textit{n}+\textit{n} five-body model}\]

MN potential for NN  
Woods-Saxon potential for N-\^{12}C  
Pauli constraint for N-\^{12}C relative motion

\[\text{CG+SVM}\]

\[\text{\^{12}C+\alpha spectroscopic amplitude}\]

Energy and \(\alpha\)-probability are well reproduced  
Extension to other \(J^{\pi}\) states …
Summary

\(^2\text{H}(d,p)^3\text{H}, ^2\text{H}(d,n)^3\text{He}, ^2\text{H}(d,\gamma)^4\text{He}\) at low energies are all well reproduced with realistic potentials by combining the methods of structure and reactions. They clearly show the sensitivity to \textbf{D-state probabilities} brought about by the tensor force.

\textbf{Triple-alpha reaction} at very low energies poses a difficult problem. A reaction rate calculation is in progress using adiabatic hyperspherical approach and complex absorbing potential. A fair prediction of \(^{12}\text{C}(\alpha,\gamma)^{16}\text{O}\) reaction rate is still difficult from the point of view of the nuclear structure of the states relevant to the process.

Many uncertainties in supernova nucleosynthesis and roles of unstable nuclei

\textit{Obrigado pela sua atenção}
A partir daí tornou-se difícil a manutenção do afluxo de físicos europeus por períodos prolongados. Nesta época o IFT foi brindado com uma série de visitas de físicos japoneses, tais como a do eminente Mituo Taketani, da Universidade de Rikkyo, em Tókio, e Yasuhisa Katayama, da Universidade de Kyoto. Dois anos depois, quando do retorno destes ao Japão, foram substituídos por Tatuoki Miyasima, da Tokyo University of Education, Daisuke Itô o da Hokkaido University, e Jun'ichi Osada, do Instituto de Tecnologia de Tokio, que também permaneceram por dois anos. Fechou-se assim um ciclo de oito anos em que o IFT consolidou seu padrão de nível internacional.
Several approaches to accurate bound-state calculations
Faddeev-Yakubovsky, GFMC, NCSM, EFHH, Variational, …

$^4$He; See e.g. Kamada et al. PRC64 (2001)

**Variational method for few-nucleon bound states**

**Correlated Gaussian (CG) basis**

For a recent review
J. Mitroy et al., RMP 85 (2013)

Pair correlation is best described with pairwise coordinates
\[ r_1, r_2, \ldots, r_N \rightarrow x_1, x_2, \ldots, x_{N-1} \]

\[
\exp \left[ -\frac{1}{2} \sum_{i<j} \left( \frac{r_i - r_j}{b_{ij}} \right)^2 \right] = \exp \left( -\frac{1}{2} \tilde{x} A \tilde{x} \right)
\]

Spherical part
\[ \tilde{x} A \tilde{x} = \sum_{i,j} A_{ij} x_i \cdot x_j \]

Global vector:
\[ u_1 x_1 + u_2 x_2 + \ldots + u_{N-1} x_{N-1} \]

\[ \mathcal{V}_{LM}(u_1 x_1 + u_2 x_2 + \ldots) \]

\[ \left[ \mathcal{V}_{L1}(u_1 x_1 + u_2 x_2 + \ldots) \mathcal{V}_{L2}(v_1 x_1 + v_2 x_2 + \ldots) \right]_{LM} \]

Combination of basis functions
\[ \Phi_{(LS)J M J, T M T}^\pi = A \left[ \phi_L \chi_S \right]_{J M J} \eta T M T \]
Each basis contains many variational parameters: $b_{ij}, A_{ij}, u_i, L, S, \ldots$
determined by stochastic variational method (SVM)

Y. S., K. Varga, Lecture Notes in Physics, m54 (Springer, 1998)

**Advantages of CG**

Matrix elements can be calculated analytically
Flexible enough to describe rapidly changing functions (asymptotics, short-range depletion)
Permutation symmetry is easily imposed
CG form is invariant under coordinate transformation

**Other type of Gaussian basis:**
Partial Wave Expansion using rearrangement channels

$$\exp(-a_1 x_1^2 - a_2 x_2^2 - a_3 x_3^2 - \ldots) \ldots [\ldots [[\psi_1(x_1) \psi_2(x_2)]L_{12} \psi_3(x_3)]L_{123} \ldots ]LM_L$$

$a_i, l_i, L, S, \ldots$
Few-body models for nuclear astrophysics

Y. Suzuki (Niigata Univ. & RIKEN)

Ab initio calculation of four-nucleon reactions

Transfer reactions $^2\text{H}(d,p)^3\text{H}, ~^2\text{H}(d,n)^3\text{He}$
Radiative capture $^2\text{H}(d,\gamma)^4\text{He}$
showing the role of tensor force

Niigata-ULB collaboration (K.Arai, S.Aoyama, Y.S., P.Descouvemont, D. Baye)

Challenges to a fair calculation

Triple-$\alpha$ reaction $\alpha+\alpha+\alpha \rightarrow ^{12}\text{C}$

Radiative capture $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$
In collaboration with W. Horiuchi

São Paulo 2015
**Reactions relevant to nuclear astrophysics**

**Transfer reactions** … nuclear interaction

**Capture reactions** … electromagnetic interaction

The latter is negligible unless the former is closed

**Transfer reactions**: continuum to continuum transition
collision (scattering) matrix $U$

$$
\sigma_t(E, i \rightarrow f) = \frac{\pi}{k^2} \sum_{J\pi} \frac{2J + 1}{(2I_1 + 1)(2I_2 + 1)} \sum_{\ell\ell'I'} |U_{i\ell I, f\ell' I'}(E)|^2
$$

**Radiative capture reactions**: continuum to bound-state transition
first-order perturbation theory

$M$: electric, magnetic multipole operator
selection rules

$$
\sigma_c(E, J_f \pi_f) \sim \sum_{\sigma\lambda J_i I \ell_i} \frac{1}{2\ell_i + 1} \frac{8\pi(\lambda + 1)}{\hbar\lambda(2\lambda + 1)!!^2} k^{2\lambda+1} \left| \langle \Psi_{J_f \pi_f} || M^g_{\lambda} || \Psi_{J_i \pi_i} (E) \rangle \right|^2
$$
Realistic vs effective NN potentials

S. Aoyama et al., FBS 52 (2012)

Two nuclei in the internal region strongly interact to change their structure

MN: a mild force
no short-range repulsion
no tensor force
reproduces low-energy data

I-IV: inclusion of pseudo states of \( d \) (\( 1^+, 0^+, 2^+, \ldots \) )
FULL: inclusion of both 2N+2N and 3N+N configurations

d+d S-wave phase shifts
Convergence of \( 0_1^+ \) and \( 0_2^+ \) states

A microscopic description of d+d scattering with realistic forces is not simple
**Microscopic R-matrix method**


Divide internal and external regions

\[ \Psi^\pi_{\text{int JM}} = \sum_i \Psi^\pi_{i JM} \quad \text{Expansion in } L^2 \text{ basis (CG)} \]

\[ \Psi^\pi_{\text{ext JM}} = \sum_\alpha g_\alpha(r_\alpha) \Phi^\pi_{\alpha JM} \quad \text{Expansion in channel wave functions} \]

\( g_\alpha \) contains \( U(S) \) matrix to be determined

\[
(H + \mathcal{L} - E) \Psi^\pi_{\text{int JM}} = \mathcal{L} \Psi^\pi_{\text{ext JM}} \quad r_\alpha \leq R
\]

\[ \Psi^\pi_{\text{int JM}} = \Psi^\pi_{\text{ext JM}} \quad \text{at } r_\alpha = R \]

\[ \mathcal{L} = \sum_\alpha \frac{\hbar^2}{2\mu_\alpha R} \Phi^\pi_{\alpha JM} \delta(r_\alpha - R) \left( \frac{\partial}{\partial r_\alpha} - \frac{b_\alpha}{r_\alpha} \right) r_\alpha \Phi^\pi_{\alpha JM} \]

Bloch operator \( \mathcal{L} \) is a surface operator:
- makes the kinetic energy Hermitean in the internal region
- makes derivatives of \( \Psi_{\text{int}} \) and \( \Psi_{\text{out}} \) at \( R \) equal
Initial d+d channel

Channel spin $I = 0, 1, 2 \ (I=1+1)$
Two-d exchange $I + l = \text{even}$
Pairy $\pi = (-1)^l$
Total angular momentum $J = I + l$

Possible channels $^{2I+1}l_J$

<table>
<thead>
<tr>
<th>channel</th>
<th>$J^\pi$</th>
<th>$0^+$</th>
<th>$1^+$</th>
<th>$2^+$</th>
<th>$0^-$</th>
<th>$1^-$</th>
<th>$2^-$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d(1^+)+d(1^+)$</td>
<td></td>
<td>$^1S_0$</td>
<td>$^5D_1$</td>
<td>$^5S_2$</td>
<td>$^3P_0$</td>
<td>$^3P_1$</td>
<td>$^3P_2$</td>
<td></td>
</tr>
<tr>
<td>$t(\frac{1}{2}^+)+p(\frac{1}{2}^+)$, $h(\frac{1}{2}^+)+n(\frac{1}{2}^+)$</td>
<td></td>
<td>$^1S_0$</td>
<td>$^3S_1$</td>
<td>$^1D_2$</td>
<td>$^3P_0$</td>
<td>$^1P_1$</td>
<td>$^3P_2$</td>
<td></td>
</tr>
</tbody>
</table>

Low-energy reactions are dominated by initial S wave
S-wave d+d entrance is possible only for $J^\pi = 0^+, 2^+$
Radiative capture $^2\text{H}(d,\gamma)^4\text{He}$ at astrophysical energy

At extreme low energy $S$-wave is predominant $\rightarrow J^{\pi}=0^+, 2^+$

$0^+ \rightarrow 0^+ EM$ transition is forbidden

$2^+ \rightarrow 0^+ EM$ transition: $E2$

If no tensor force $\rightarrow$ No $D$ states in $d$ and $^4\text{He}$

$\rightarrow$ No $S$-wave $E2$ transition

but $D$-wave transition

AV8' reproduces $S$-factor,

but MN fails at $E < 0.3\text{MeV}$

Other possible transitions: $E1$ ($^3\text{P}_1$), $M1$ ($^5\text{D}_1$)

$d + d \rightarrow ^4\text{He}$

$T=0 \quad T=0$

Isospin conservation

$E1$ operator

$D = e \sum_{i\in \text{proton}} (r_i - R) \approx 0$

Isospin mixing due to Coulomb force